Math 104A - Homework 3. Due Friday, April 15.

1. Let \( R = \mathbb{Z}[\sqrt{-41}] \). Show that 3 is irreducible but not prime in \( R \). Is \( R \) an Euclidean domain?

2. Show that \( \mathbb{Z}[-2] = \{ a + b\sqrt{-2} : a, b \in \mathbb{Z} \} \) is an Euclidean domain.

   \textit{Hint: You may want to look at the proof that} \( \mathbb{Z}[\sqrt{-1}] \text{ is an Euclidean domain.} \)

3.

   (i) Show that if \( x \) is an odd integer, then \( x^2 \equiv 1 \mod 8 \). Conclude that if \( x \) is any integer, then \( x^2 \equiv 0, 1 \) or \( 4 \mod 8 \).

   (ii) Show that if \( x \) is any integer, then \( x^2 \equiv 0 \) or \( 1 \mod 3 \).

   (iii) From (i) and (ii), show that if \( p \geq 5 \) is a prime, then \( p^2 \equiv 1 \mod 24 \).

4.

   (i) Show that if \( p \equiv 3 \mod 4 \) is any integer, then the equation \( x^2 + y^2 = p \) has no integer solutions. You may wish to use problem 3(i).

   (ii) Show that if \( p \equiv 7 \mod 8 \) is any integer, then the equation \( x^2 + y^2 + z^2 = p \) has no integer solution. You may wish to use problem 3(i).

   \textit{Remark: By contrast, it can be shown that if} \( p \) \text{ is any positive integer, the equation} \( x^2 + y^2 + z^2 + w^2 = p \text{ always has integer solutions.} \)

5. Let \( p \equiv 3 \mod 4 \) be a prime integer, and regard \( p = p + 0\sqrt{-1} \) as an element of the Gaussian integers \( R = \mathbb{Z}[\sqrt{-1}] \). Using Problem 4(i), show that \( p \) is irreducible, hence prime, in \( R = \mathbb{Z}[\sqrt{-1}] \).

   \textit{Hint: You should proceed in the usual way, assuming} \( p = \alpha \cdot \beta \text{ in} R \text{ and taking norms.} \)

   \textit{Remark: By contrast, if} \( p \equiv 1 \mod 4 \text{ is a prime integer, it can be shown that} p \text{ is not prime in} \mathbb{Z}[\sqrt{-1}] \).

6.

   (i) Write down the multiplication table in \( \mathbb{Z}_9 \). From the table, read off the units in this ring together with their inverses.

   (ii) Write down the units in the ring \( \mathbb{Z}_{24} \) without constructing the multiplication table.

   (iii) Find the inverse of the units 7 and 11 in the ring \( \mathbb{Z}_{120} \) without constructing the multiplication table.