1. Show directly from the definition that
\[ \sin(2z) = 2 \sin z \cos z \]

Solution:
\[ \sin(2z) = \frac{e^{2zi} - e^{-2zi}}{2i} = 2 \frac{(e^{zi} - e^{-zi})}{2i} \frac{e^{zi} + e^{-zi}}{2} = 2 \sin z \cos z \]

2. Write the following complex numbers in standard form:
   (i) \((-1 + i\sqrt{3})^i\). What is the principal value?
   (ii) \(\tan^{-1}(2i)\),
   (iii) \(\tan\left(\frac{i\pi}{2}\right)\),
   (iv) solve the equation \(\sin z = 2\).

Solution:
   (i) \((-1 + i\sqrt{3})^i = e^{i(\ln(2) + i(\frac{2\pi}{3} + 2n\pi))} = e^{-\frac{2\pi}{3} + 2n\pi} \cos(\ln 2) + ie^{-\frac{2\pi}{3} + 2n\pi} \sin(\ln 2)\).
   P.V. \((-1 + i\sqrt{3})^i = e^{-\frac{2\pi}{3}} \cos(\ln 2) + ie^{-\frac{2\pi}{3}} \sin(\ln 2)\).
   (ii) \(\tan^{-1}(2i) = z \iff \tan(z) = 2i \iff \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -2 \iff e^{iz} - e^{-iz} = -2e^{iz} - 2e^{-iz} \iff e^{-iz} = -3e^{iz} \iff e^{-2iz} = -3 \iff z = -\frac{\pi}{2} + \frac{i \ln 3}{2} + n\pi\).
   (iii) \(\tan\left(\frac{i\pi}{2}\right) = \frac{e^{i\left(\frac{i\pi}{2}\right)} - e^{-i\left(\frac{i\pi}{2}\right)}}{i(e^{i\frac{i\pi}{2}} + e^{-i\left(\frac{i\pi}{2}\right)})} = i \cdot \frac{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}}{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}\).
   (iv) \(\sin z = 2 \iff e^{iz} - e^{-iz} = 4i \iff w - \frac{1}{w} = 4i, \text{ with } w = e^{iz} \iff w^2 - 4iw - 1 = 0 \iff w = (2 \pm \sqrt{3})i \iff e^{iz} = (2 \pm \sqrt{3})e^{\frac{\pi}{2}} \iff z = \frac{\pi}{2} + 2k\pi + i\ln(2 \pm \sqrt{3})\).
3. Evaluate the following integrals by parametrizing the contour:
   (i) \( \int_C x\,dz \) where \( C \) is the oriented line segment joining 1 to \( i \),
   (ii) \( \int_C (z - 1)\,dz \) where \( C \) is the semicircle joining 0 to 2,
   (iii) \( \int_C \cos(\frac{z}{2})\,dz \) where \( C \) is the line segment joining 0 to \( \pi + 2i \).

Solution:
   (i) Parametrize the contour by \( z = (i - 1)t + 1 \) with \( 0 \leq t \leq 1 \). Then
   \[
   \int_C x\,dz = \int_0^1 (1 - t)(i - 1)\,dt = \frac{i - 1}{2}.
   \]
   (ii) Using the parametrization \( z = 1 + e^{-i\theta}, \quad -\pi \leq \theta \leq 0 \), \( dz = -ie^{-i\theta}\,d\theta \),
   we compute
   \[
   \int_C (z - 1)\,dz = \int_{-\pi}^0 e^{-i\theta} (-ie^{-i\theta})\,d\theta = -i \int_{-\pi}^0 e^{-2i\theta}\,d\theta = \frac{1}{2}e^{-2i\theta} \bigg|_{\theta=-\pi}^{\theta=0} = 0.
   \]
   (iii) Writing \( z = (\pi + 2i)t \) for \( 0 \leq t \leq 1 \), we compute
   \[
   \int_C \cos(\frac{z}{2})\,dz = \frac{\pi + 2i}{2} \int_0^1 e^{i(\frac{\pi}{2}+i)t} + e^{-i(\frac{\pi}{2}+i)t}\,dt = -i \left(e^{i(\frac{\pi}{2}+i)t} - e^{-i(\frac{\pi}{2}+i)t}\right) \bigg|_{t=0}^{t=1} = e + e^{-1}.
   \]

4. Evaluate
   \[
   \int_C z^{-1+i}\,dz
   \]
   where \( C \) is the positively oriented unit circle, and the integrand is defined by choosing the branch
   \( 0 < \arg(z) < 2\pi \).
   What happens if we take \( -\pi < \arg(z) < \pi \)?

   Solution: For \( z = e^{i\theta} \), we have
   \[
   \int_C z^{-1+i}\,dz = \int_0^{2\pi} e^{i\theta(i-1)} \cdot ie^{i\theta}\,d\theta = i \int_0^{2\pi} e^{-\theta}\,d\theta = i(1 - e^{-2\pi}).
   \]
   If we take \( -\pi < \arg(z) < \pi \) we’ll get \( i(e^\pi - e^{-\pi}) \).

5. Compute
   \[
   \int_C z^a\,dz
   \]
   where \( C \) is the counterclockwise unit circle and \( a \) is any real number. The principal value is used for the integrand. Do it in two ways: by picking a parametrization of \( C \), and by using a suitable anti-derivative.
Solution: Let $z = e^{i\theta}$. If $a \neq -1$, we have

\[ \int_C z^a \, dz = \int_{-\pi}^{\pi} e^{ia\theta} \cdot ie^{i\theta} \, d\theta = \frac{e^{i\pi(a+1)} - e^{-i\pi(a+1)}}{a+1} = \frac{2i}{a+1} \sin((a+1)\pi). \]

If $a = -1$, then

\[ \int_C z^{-1} \, dz = \int_{-\pi}^{\pi} i \, d\theta = 2\pi i. \]

For the second method, let us assume first $a \neq -1$. Note that the principal value of $z^a = e^{a\log z}$ is undefined at the negative reals. To compute the integral, we will split the circle into the upper and lower halves. For the integral across the upper half $C_1$, we will replace $z^a$ by a different branch which is everywhere defined and holomorphic. We pick a branch cut which doesn’t cross $C_1$, for instance

\[-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}.

This branch of $z^a$ agrees with the principal branch along $C_1$. Then,

\[ \int_{C_1} z^a \, dz = \frac{e^{(a+1)\log(z=-1)} - e^{(a+1)\log(z=1)}}{a+1} = \frac{e^{\pi i(a+1)} - 1}{a+1} \]

For the integral along $C_2$, we may branch cut along a half-line which avoids $C_2$, for instance

\[ -\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}. \]

The values of $z^a$ for this branch coincide with the principal values along $C_2$. We evaluate

\[ \int_{C_2} z^a \, dz = \frac{e^{(a+1)\log(z=1)} - e^{(a+1)\log(z=-1)}}{a+1} = \frac{1 - e^{-\pi i(a+1)}}{a+1}. \]

Putting things together

\[ \int_C z^a \, dz = \frac{e^{\pi i(a+1)} - e^{-\pi i(a+1)}}{a+1} \]

just as before. The case $a = -1$ is done in similar way but using the antiderivative of $\frac{1}{z}$ which is $\log(z)$ and was done in class.

**N.B.** You should be careful when choosing your branch cut. It is especially easy to get a wrong answer when evaluating the integral along $C_2$. Picking a branch cut which avoids $C_2$ is not enough, you have to make sure that the chosen branch coincides with the principal value, so that you are not changing the integral. For instance the branch cut

\[ \frac{\pi}{2} < \arg(z) < \frac{5\pi}{2} \]

would not work in this case. This is because this branch of $z^a$ does not agree with the principal value along $C_2$. This can be seen, for instance, at the
point 1. There, the branch (2) gives the value $1^{a+1} = e^{2\pi i(a+1)}$ which differs from the principal value $1^{a+1} = 1$. However, you can convince yourself that the chosen branch (1) does work.

6. Show that

$$\int_{-1}^{1} z^i \, dz = \frac{(1 + e^{-\pi})(1 - i)}{2}$$

for any path joining $-1$ to $1$ which lies above the real axis, endpoints excluded. The principal value is used for the integrand. Do it also for any path joining $-i$ and $i$ which lies on the right of the imaginary axis.

*Solution:* Observe that the integrand does not exist at the endpoint $z = -1$. To fix this, let us consider a different branch cut along $-\pi < \arg(z) < 3\pi/2$.

The branch of $z^i$ above agrees along the path of integration with the principal branch (except possibly at $z = -1$ where the latter is undefined). Therefore, we may safely work with the new branch considered above. Now, note that $z^i$ has an antiderivative in the upper half plane given by

$$\frac{z^{i+1}}{i + 1}$$

where the new branch is used again in the definition of $z^{i+1}$. Therefore

$$\int_{-1}^{1} z^i \, dz = \frac{e^{(i+1)\log(z=1)} - e^{(i+1)\log(z=-1)}}{i + 1} = \frac{1 - e^{(i+1)(i\pi)}}{i + 1} = \frac{(1 + e^{-\pi})(1 - i)}{2}.$$

The integral along the second path causes no problems since the path does not intersect the branch cut at the negative reals. Therefore,

$$\int_{-i}^{i} z^i \, dz = \frac{e^{(i+1)\log(z=i)} - e^{(i+1)\log(z=-i)}}{i + 1} = \frac{e^{(i+1)i\pi} - e^{(i+1)-i\pi}}{i + 1} = \frac{(e^{-\pi} + e^{\pi})(i + 1)}{2}.$$

7. What are the values of the following integrals:

(i) $\int_C \frac{z^2}{z^2 - 3} \, dz$ where $C$ is the positively oriented unit circle,

(ii) $\int_C \log(z + 2) \, dz$ where $C$ is the positively oriented unit circle.

*Solution:* In both cases the integrands are holomorphic on and inside the unit circle so by Cauchy Theorem the integrals are 0. For the first function, the pole is at $z = 3$ which is clearly outside $C$. The second function is holomorphic everywhere except for the line $z = -2 + x$ with $x$ is a negative real. This line also avoids the unit circle.
8. Determine the value of the integral
\[ \int_C (z - 1)^n \, dz \]
where \( n \) is any integer and \( C \) is a positively oriented square of side \( a \), which doesn’t go through 1.

Solution: If 1 is not inside the square, the integral is 0 because \((z - 1)^n\) is holomorphic inside \( C \) no matter whether \( n \) is positive, negative or 0.

If 1 is inside the square, and \( n \geq 0 \) the same reasoning shows that the integral is 0.

So let us consider the case when \( n < 0 \), in which case the function \((z - 1)^n\) has a pole at \( z = 1 \). We can consider a small circle around 1 that lies inside the square. Since \((z - 1)^n\) is holomorphic in the area between the square and the circle, the integrals over these curves are equal. Consider parametrization of the circle \( z = 1 + re^{i\theta} \), where \( r \) is the radius. Then
\[
\int_C (z - 1)^n \, dz = r^{n+1} \int_0^{2\pi} e^{in\theta} \cdot ie^{i\theta} \, d\theta = \frac{2ir^{n+1}}{n+1} e^{i(n+1)\theta} \bigg|_{\theta=0}^{\theta=2\pi} = 0
\]
The case \( n = -1 \) is special since then the denominator becomes 0. The integral can be computed by hand to be \( 2\pi i \).

9. Show that the area enclosed by a positively oriented simple closed curve \( C \) is given by
\[ \frac{1}{2i} \int_C \bar{z} \, dz \]

Solution: We have
\[ \int_C \bar{z} \, dz = \int_C (x - iy) \cdot (dx + idy) = \int_C (x \, dx + y \, dy) + i(x \, dy - y \, dx). \]
Let \( R \) be the region enclosed by \( C \). We can apply Green’s theorem to each of the two terms above to conclude
\[ \int_C \bar{z} \, dz = \int \int_R 2i \, dx \, dy = 2\pi \text{ area (} R \text{)}. \]