1. Let $C_1$ denote the positively oriented circle $|z| = 4$ and $C_2$ the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. With the aid of Cor. 2 in Sec 46, point out why
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \]
when
(a) $f(z) = \frac{1}{3z^2 + 1}$  (b) $f(z) = \frac{z + 2}{\sin \frac{z}{2}}$  (c) $f(z) = \frac{z}{1 - e^z}$

Solution: We need to show the given functions are holomorphic in the area between and on the two curves, call it $A$.

For (a), the zeros of $3z^2 + 1$ are $\pm \frac{i}{\sqrt{3}}$, they both lie in the area enclosed by $C_1$, so the function is holomorphic in $A$. For part (b), we observe that $\sin \frac{z}{2} = 0 \Leftrightarrow z = 2k\pi \Rightarrow f(z)$ is holomorphic in $A$. Finally, for (c), the denominator is zero when $1 - e^z = 0 \Leftrightarrow z = 2ik\pi \Rightarrow f(z)$ is holomorphic in $A$.

2. Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$. Evaluate each of these integrals:

(a) $\int_C \frac{e^{-z}}{z - \frac{i\pi}{2}} \, dz$,  (b) $\int_C \frac{\cos z}{z(z^2 + 8)} \, dz$,  (c) $\int_C \frac{z}{2z + 1} \, dz$,
(d) $\int_C \frac{\cosh z}{z^4} \, dz$,  (e) $\int_C \frac{\tan \frac{z}{2}}{z - x_0} \, dz, -2 < x_0 < 2$.

Solution: We use the Cauchy’s integral formula.

(a) $\int_C \frac{e^{-z}}{z - \frac{i\pi}{2}} \, dz = 2i\pi e^{-\frac{i\pi}{2}} = 2\pi$

(b) $\int_C \frac{\cos z}{z(z^2 + 8)} \, dz = 2i\pi \frac{\cos(0)}{0^2 + 8} = \frac{i\pi}{4}$

(c) $\int_C \frac{z}{2z + 1} \, dz = \int_C \frac{z/2}{z + 1/2} \, dz = 2i\pi \cdot \frac{-1}{4} = -\frac{i\pi}{2}$

(d) $\int_C \frac{\cosh z}{z^4} \, dz = 2i\pi \frac{\sinh(0)}{3!} = 0$

(e) $\int_C \frac{\tan \frac{z}{2}}{z - x_0} \, dz = 2i\pi \cdot \frac{1}{2\cos^2 \frac{x_0}{2}} = \frac{i\pi}{\cos^2 \frac{x_0}{2}}$
3. Find the value of the integral of \( g(z) \) around the circle \(|z - i| = 2\) in the positive sense when
\[
(a) \ g(z) = \frac{1}{z^2 + 4}, \quad (b) \ g(z) = \frac{1}{(z^2 + 4)^2}
\]

Solution:
\[
(b) \ \int_{|z-i|=2} \frac{1}{(z^2 + 4)} dz = \int_{|z-i|=2} \frac{1}{(z-2i)(z+2i)} dz = 2i\pi \cdot \frac{1}{(2i + 2i)} = \frac{\pi}{2}
\]
\[
(a) \ \int_{|z-i|=2} \frac{1}{z^2 + 4} dz = \int_{|z-i|=2} \frac{1}{(z-2i)(z+2i)} dz = 2i\pi \cdot \frac{2}{(2i + 2i)^3} = \frac{\pi}{16}
\]

4. Let \( C \) be the circle \(|z| = 3\), described in the positive sense. Show that if \( g \) is analytic within and on a simple closed contour \( C \), then \( g(2) = 8i\pi \). What is the value of \( g(w) \) when \(|w| > 3\).

Solution:
\[
g(2) = \left| \int_{C} \frac{2z^2 - z - 2}{z - 2} \ dz = 2i\pi(2z^2 - z - 2) \right|_{z=2} = 8i\pi
\]
When \(|w| > 3\), the integrand is holomorphic on and inside \( C \) (since \( z = w \) cannot happen on the circle \( C \)), therefore it is zero.

5. Show that if \( f \) is analytic within and on a simple closed contour \( C \) and \( z_0 \) is not on \( C \), then
\[
\int_{C} \frac{f'(z)}{z - z_0} \ dz = \int_{C} \frac{f(z)}{(z - z_0)^2} \ dz
\]

Solution: \( f \) holomorphic implies \( f' \) holomorphic inside and on \( C \). If \( z_0 \) is outside \( C \), both sides will be zero since the integrands will be holomorphic inside and on \( C \). If \( z_0 \) is inside \( C \), the left-hand side will equal to \( 2i\pi f'(z_0) \) by Cauchy formula for \( f' \). By the formula for derivatives, the right-hand side will be \( 2i\pi f'(z_0) \) also.

6. Let \( f \) be an entire function such that \(|f(z)| \leq A|z|\) for all \( z \), where \( A \) is a fixed positive number. Show that \( f(z) = az \) where \( a \) is a complex number.

Solution: Let \( z_0 \in \mathbb{C} \) and \( C_R \) be the circle of radius \( R \) around \( z_0 \). Then, the maximum value of \(|f(z)|\) on \( C_R \) can be estimated
\[
|f(z)| \leq A|z| \leq A(R + |z_0|).
\]
Just as in the proof of the Liouville’s theorem, we have
\[
|f''(z_0)| = \left| \frac{1}{i\pi} \int_{C_R} \frac{f(z)}{(z - z_0)^3} \ dz \right| \leq \frac{1}{\pi} \cdot \frac{A(|z_0| + R)}{R^3} \cdot 2\pi R = \frac{2A(|z_0| + R)}{R^2}
\]
for every \( R \). Making \( R \to \infty \), we obtain \( f''(z_0) = 0 \). Since \( z_0 \) was arbitrary, we get \( f''(z) \equiv 0 \), hence \( f(z) = az + b \), where \( a, b \) are complex constants. From \(|f(z)| \leq A|z|\) when \( z = 0 \) we get \( f(0) = 0 \), so \( b = 0 \). Therefore \( f(z) = az \).
7. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = Re[f(z)]$ has an upper bound $u_0$. Show $u(x, y)$ must be a constant.

**Solution:** The function $g(z) = e^{f(z)}$ is entire, and

$$|g(z)| = |e^{u+i\theta}| = |e^u| = |e^{u}| \leq e^{u_0}.$$ 

By Liouville’s theorem $g(z)$ is constant so $g'(z) = 0$. Now, $g'(z) = e^{f(z)}f'(z)$ so $f'(z) = 0$ since the exponential cannot be 0, implying that $f(z)$ is constant.