Problem 1 [a] $\sqrt{3}+i = 2e^{\frac{i\pi}{6}} = e^{\ln 2 + \frac{i\pi}{6}} \Rightarrow (\sqrt{3}+i)^{-i} = e^{-i\ln 2 + \frac{\pi}{6}} = \boxed{e^{+\frac{\pi}{6}}\cos(\ln 2) - i\, e^{+\frac{\pi}{6}}\sin(\ln 2)}$

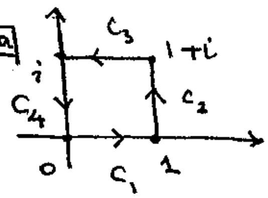
[b] $\cot^{-1}(3i) = z \Rightarrow \cot z = 3i \Rightarrow \dfrac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}} = 3 \Rightarrow e^{iz}+e^{-iz} = 3e^{iz}-3e^{-iz} \Rightarrow 2e^{iz}-4e^{-iz}=0$

$\Rightarrow e^{2iz} = 2 \Rightarrow 2iz = \ln 2 + 2k\pi i \Rightarrow z = -\frac{i}{2}\ln 2 + k\pi$. Principal value $\boxed{\cot^{-1}(3i) = -\frac{i\ln 2}{2}}$

[c] $\sin\left(2i+\frac{\pi}{4}\right) = \dfrac{e^{i\left(2i+\frac{\pi}{4}\right)} - e^{-i\left(2i+\frac{\pi}{4}\right)}}{2i} = \dfrac{e^{-2}\cdot e^{\frac{i\pi}{4}} - e^{2}\cdot e^{-\frac{i\pi}{4}}}{2i} = \dfrac{e^{-2}(1+i) - e^{2}(1-i)}{2i\sqrt{2}} =$

$= \boxed{\dfrac{e^{-2}+e^{2}}{2\sqrt{2}} + i\cdot\dfrac{e^{2}-e^{-2}}{2\sqrt{2}}}$

Problem 2 [a]



along C_1: $z = t,\ 0\le t\le 1$. $\displaystyle\int_{C_1}\bar{z}\,dz = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$.

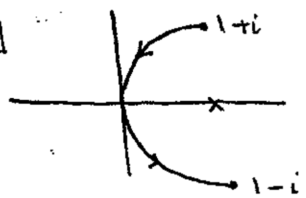
along C_2 $\quad z = 1+it,\ 0\le t\le 1$. $\displaystyle\int_{C_2}\bar{z}\,dz = \int_0^1 (1-it)\cdot i\,dt = i + \frac{1}{2}$.

along C_3 $\quad z = 1-t+i,\ 0\le t\le 1$, $\displaystyle\int_{C_3}\bar{z}\,dz = \int_0^1 ((1-t)-i)(-dt) = i - \frac{1}{2}$.

along C_4: $z = i(1-t),\ 0\le t\le 1$. $\displaystyle\int_{C_4}\bar{z}\,dz = \int_0^1 -i(1-t)\cdot(-i\,dt) = -\int_0^1 (1-t)\,dt = -\frac{1}{2}$.

$\displaystyle\Rightarrow \int_C f(z)\,dz = \int_{C_1}\bar{z}\,dz + \int_{C_2}\bar{z}\,dz + \int_{C_3}\bar{z}\,dz + \int_{C_4}\bar{z}\,dz = \frac{1}{2} + \left(i+\frac{1}{2}\right) + \left(i-\frac{1}{2}\right) - \frac{1}{2} = \boxed{2i}$

[b]



$z = 1+e^{i\theta} \Rightarrow dz = ie^{i\theta}$, $\sqrt{z-1} = e^{i\theta/2}$. For the principal part θ goes from $\left(\frac{\pi}{2},\pi\right)$ and $\left(-\pi,-\frac{\pi}{2}\right)$.

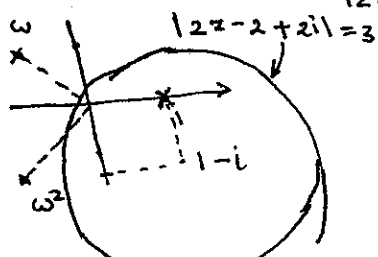
$\displaystyle\int_C \sqrt{z-1}\,dz = i\left(\int_{\pi/2}^{\pi} e^{3i\theta/2}\,d\theta + \int_{-\pi}^{-\pi/2} e^{\frac{3i\theta}{2}}\,d\theta\right) = \frac{2}{3}\,e^{\frac{3i\theta}{2}}\Big|_{\theta=\frac{\pi}{2}}^{\theta=\pi} +$

$\displaystyle + \frac{2}{3}\,e^{\frac{3i\theta}{2}}\Big|_{\theta=-\pi}^{\theta=-\frac{\pi}{2}} = \frac{2}{3}\Big(\underset{-i}{\underset{\shortparallel}{e^{\frac{3i\pi}{2}}}} - e^{\frac{3i\pi}{4}} + e^{-\frac{3i\pi}{4}} - \underset{i}{\underset{\shortparallel}{e^{-\frac{3\pi i}{2}}}}\Big) = \frac{2}{3}\left(-2i + 2\sin\left(-\frac{\pi}{4}\right)\cdot i\right) =$

$= \boxed{-\dfrac{2(2+\sqrt{2})}{3}\,i}$

Problem 3 [a] $\displaystyle\int_{|z-i|=2}\frac{dz}{z^4-2z^3} = \int_{|z-i|=2}\frac{\frac{1}{z-2}}{z^3}\,dz = \frac{2\pi i}{2!}\left(\frac{1}{z-2}\right)''\Big|_{z=0} = \pi i\cdot\frac{2}{(z-2)^3}\Big|_{z=0} = \boxed{-\frac{\pi i}{4}}$.

[b] $z^3 = 1$.



The only root of $z^3=1$ inside the circle $|2z-2+2i|=3$ is $z=1$. Thus by Cauchy's formula

$\displaystyle\int_C\frac{dz}{z^3-1} = \int\frac{1}{z^2+z+1}\cdot\frac{1}{z-1}\,dz = 2\pi i\cdot\frac{1}{z^2+z+1}\Big|_{z=1} = \boxed{\frac{2\pi i}{3}}$.

[c] $z^2+4iz+1 = 0 \Rightarrow z = -2i \pm\sqrt{(2i)^2-1} = (-2\pm\sqrt{5})i$. Only $z_0 = (-2+\sqrt{5})i$ has $|z_0|<1$,

$z_1 = (-2-\sqrt{5})i$ has $|z_1|>1$. By Cauchy:

$\displaystyle\int_{|z|=1}\frac{dz}{z^2+4iz+1} = \int_{|z|=1}\frac{dz}{(z-z_0)(z-z_1)} = 2\pi i\cdot\frac{1}{z_0-z_1} = 2\pi i\cdot\frac{1}{2i\sqrt{5}} = \boxed{\frac{\pi}{\sqrt{5}}}$

Problem 4 $f(z) = e^{z} + i e^{-z} = e^{x+iy} + i e^{-x-iy} = e^{x}(\cos y + i \sin y) + i e^{-x}(\cos y - i \sin y) \Rightarrow$

$\Rightarrow |f(z)|^2 = (e^{x}\cos y + e^{-x}\sin y)^2 + (e^{x}\sin y + e^{-x}\cos y)^2 = e^{2x}(\cos^2 y + \sin^2 y) + e^{-2x}(\sin^2 y + \cos^2 y)$

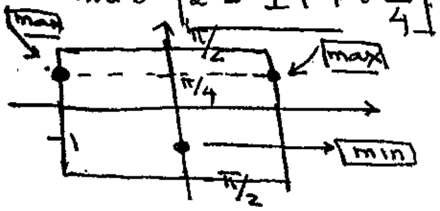
$+ 4\cos y \sin y = e^{2x} + e^{-2x} + 2\sin 2y$. Maximum: $\sin(2y) = 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ so

$\boxed{y = \frac{\pi}{4}}$. Also $F(x) = e^{2x} + e^{-2x}$ has to be maximal. $F'(x) = 2(e^{2x} - e^{-2x}) = 0 \Rightarrow$

$\Rightarrow x = 0$ is a critical point. & we need to check the endpoints $x = \pm 1$ as well.

In fact $F(\pm 1) = e^{2} + e^{-2} > F(0) = 2$ so $\boxed{x = \pm 1}$ are the maxima, while $x = 0$ is the min

thus $\boxed{z = \pm 1 + i\frac{\pi}{4}}$ which is on the boundary. $|f| = \left| e^{1 + \frac{i\pi}{4}} + i e^{-1 - i\frac{\pi}{4}} \right| =$



$= \left| e\left(\frac{1+i}{\sqrt{2}}\right) + i e^{-1}\left(\frac{1-i}{\sqrt{2}}\right) \right| = \left| (e + e^{-1}) \cdot \frac{1+i}{\sqrt{2}} \right| = \boxed{e + e^{-1}}$ = max value

For the minimum, we need $x = 0$ (see above) and $\sin 2y = -1 \Rightarrow y = -\frac{\pi}{4} \Rightarrow$

$\Rightarrow \boxed{z = -\frac{i\pi}{4}}$ (it is in the interior).

Problem 5 [a] If $|z| \le 1$ then $|z^5 + iz| \le |z|^5 + |iz| \le 1^5 + |i| \cdot 1 = 2 \Rightarrow z^5 + iz \neq 4$. Thus all the roots of $z^5 + iz - 4$ have $|z| > 1$. $\Rightarrow \frac{1}{z^5 + iz - 4}$ is holomorphic inside the unit disc

[b] By Cauchy $\int_{|z|=1} \frac{dz}{z^5 + iz - 4} = \boxed{0}$.

Problem 6. The function $g(z) = f(z)^2$ is holomorphic and $|g(z)| = |f(z)|^2 \le |z|$. thus, g satisfies the hypothesis of HWK 5, problem 6. Then $g(z) = az$ for some constant a.

$\Rightarrow f(z) = \sqrt{az}$. However, $\sqrt{z}$ is not continuous!! so for f to be continuous (entire) we need $a = 0 \Rightarrow$

$\Rightarrow \boxed{f(z) = 0}$ (There are other ways to obtain the answer, for instance using Taylor expansion)

Problem 7 Let $f = u + iv$. Let $g(z) = e^{(1+i)f(z)} = e^{(1+i)(u+iv)} = e^{u-v+i(u+v)} = e^{u-v}(\cos(u+v) + i\sin(u+v))$

$\Rightarrow |g(z)| = e^{u-v}$. Since $u \le v \Rightarrow u - v \le 0 \Rightarrow |g(z)| \le 1 \Rightarrow g$ is constant.

$\Rightarrow g'(z) = 0$. But $g'(z) = (1+i) f'(z) \cdot \underbrace{e^{(1+i)f(z)}}_{\neq 0} = 0 \Rightarrow f'(z) = 0 \Rightarrow \boxed{f \text{ is constant}}$

Take $f(z) = e^{z}$, $z = \frac{\pi}{2}$ for example. $\operatorname{Im} f = 0$, $\operatorname{Re} f = e^{\pi/2} > \operatorname{Im} f$ at this point.

Problem 8 [a] Since $|f(z) - 1| < 1 \Rightarrow f(z)$ is not 0 or a negative real $\Rightarrow \log f(z) := g(z)$ is well defined and holomorphic. $\Rightarrow f = e^{g}$. Then $\frac{f'}{f} = \frac{e^{g} g'}{e^{g}} = g'$. $\Rightarrow \int_c \frac{f'}{f} = \int_c g' = 0$ by the $\boxed{\text{F.T.C}}$ (c = closed).

[b] and [c]. For both, the answer is $\boxed{\text{yes}}$. Since f is holomorphic, f' is holomorphic $\Rightarrow$

$\Rightarrow \frac{f'}{f}$ is holomorphic in Ω, hence in the area enclosed by c. (For [b], the area enclosed by c is contained in Ω, Ω being simply connected ~~For [c] this is given~~). Thus $\int_c \frac{f'}{f} = 0$ by Cauchy.

[d] Take $\boxed{h(z) = \frac{1}{z}}$, $\boxed{\Omega = \mathbb{C} \setminus \{0\}}$ $\boxed{c = \text{unit circle}}$ $\oint \frac{h'}{h} = \oint \frac{-1}{z} dz = -2\pi i \neq 0$.