Problem 1. [10 points]

Determine the principal values of the following complex numbers, and write them in standard form:

(i) \((\sqrt{3} + i)^{-i}\),
(ii) \(\cot^{-1}(3i)\),
(iii) \(\sin(2i + \frac{\pi}{4})\).

Problem 2. [10 points]

(i) Let \(C\) be the unit square with vertices at 0, 1, 1 + i and i, oriented counterclockwise. Using a suitable parametrization, compute \(\int_C \bar{z} \, dz\).

(ii) Let \(C\) be the unit half circle which joins 1 + i to 1 − i with the counterclockwise orientation. Using a suitable parametrization of \(C\), compute the value of the integral
\[ \int_C \sqrt{z - 1} \, dz. \]
The principal value is used for the integrand.

Problem 3. [10 points]

Find the values of the following integrals, when the contours are oriented counterclockwise:

(i) \(\int_{|z-i|=2} \frac{dz}{z^4 - 2z^3}\).
(ii) \(\int_{|2z-2+2i|=3} \frac{dz}{z^3 - 1}\).
(iii) \(\int_{|z|=1} \frac{dz}{z^2 + 4iz + 1}\).

Problem 4. [10 points]

Consider the holomorphic function
\[ f(z) = e^z + ie^{-z} \]
defined on the closed rectangle whose corners are \(\pm 1 \pm \frac{\pi}{2} i\). Determine the maximum value of \(|f|\) (write the answer in the simplest possible form), and confirm that it is obtained on the boundary. Where does the minimum occur?
Problem 5. [5 points]

The triangle inequality from the beginning of the term (see also page 10 of the textbook) compares the absolute value of the sum with the sum of the absolute values. Precisely, for any two complex numbers \(a, b\):
\[
|a + b| \leq |a| + |b|.
\]

(i) Using the triangle inequality, explain why the complex roots of the polynomial \(z^5 + iz - 4\) must satisfy \(|z| > 1\. Hint: The roots must satisfy \(z^5 + iz = 4\).

(ii) Determine
\[
\int_C \frac{dz}{z^5 + iz - 4}
\]
where \(C\) is the counterclockwise unit circle around the origin.

Problem 6. [10 points]

Find all entire functions \(f\) which satisfy
\[
|f(z)|^2 \leq |z|.
\]

Problem 7. [10 points]

Let \(f\) be an entire function such that
\[
\text{Re } f(z) \leq \text{Im } f(z) \text{ for all } z \in \mathbb{C}.
\]
Show that \(f\) is constant. You may want to consider the function
\[
g(z) = \exp((1 + i)f(z)).
\]
Furthermore, pick your favorite example of a nonconstant entire function, which is not a linear function of the type \(az + b\), and check that the above inequality cannot hold for all \(z \in \mathbb{C}\).

Problem 8. [10 points]

Let \(f(z)\) be a holomorphic function defined in some region \(\Omega\) of the complex plane, and let \(C\) be any simple closed curve in \(\Omega\).

(i) Assume that
\[
|f(z) - 1| < 1 \text{ for all } z \in \Omega.
\]
Explain why the function \(f(z)\) can be written in the form
\[
f(z) = e^{g(z)}
\]
for some function \(g\) which is holomorphic in \(\Omega\). Conclude that
\[
\int_C \frac{f'(z)}{f(z)} \, dz = 0.
\]

(ii) Assume now only that \(f(z) \neq 0\), but that \(\Omega\) is simply connected. Is it still true that
\[
\int_C \frac{f'(z)}{f(z)} \, dz = 0?
\]
Why/why not?
(iii) Assume now only that $f(z) \neq 0$, but that $\Omega$ is not necessarily simply connected. Assume in addition that the area enclosed by $C$ is contained in $\Omega$. Is it still true that
$$
\int_C \frac{f'(z)}{f(z)} \, dz = 0?
$$
Why/why not?

(iv) Give an example of a holomorphic function $h$, of a region $\Omega$, and of a simple closed curve $C$, such that $h(z) \neq 0$ for all $z \in \Omega$, but
$$
\int_C \frac{h'(z)}{h(z)} \, dz \neq 0.
$$