

1] Using Cauchy-Riemann:

$$\begin{aligned} u_x = 2x - 2y, \quad v_y = -2y + 2x &\Rightarrow u_x = v_y \\ u_y = -2y - 2x, \quad v_x = 2x + 2y &\Rightarrow u_y = -v_x \end{aligned} \Rightarrow f \text{ entire.}$$

$$f'(z) = u_x + i v_x. \text{ For } z=i, x=0, y=1, u_x = -2, v_x = 2 \Rightarrow \boxed{f'(i) = -2 + 2i}$$

2] $-1 - i\sqrt{3} = e^{-\frac{2\pi i}{3}}$ (for principal parts we need the angle in $(-\pi, \pi)$). \Rightarrow

$$\Rightarrow \frac{e}{2}(-1 - i\sqrt{3}) = e^{1 - \frac{2\pi i}{3}} \Rightarrow \left[\frac{e}{2}(-1 - i\sqrt{3}) \right]^{3\pi i} = e^{3\pi i \left(1 - \frac{2\pi i}{3}\right)} = e^{3\pi i + 2\pi^2} = \boxed{-e^{2\pi^2}}$$

$$\begin{aligned} \text{3] } u_x &= -(e^y + e^{-y}) \sin x + (e^y - e^{-y}) \cos x & u_y &= (e^y - e^{-y}) \cos x + (e^y + e^{-y}) \sin x \\ u_{xx} &= -(e^y + e^{-y}) \cos x - (e^y - e^{-y}) \sin x & u_{yy} &= (e^y + e^{-y}) \cos x + (e^y - e^{-y}) \sin x \end{aligned}$$

$$\Rightarrow \boxed{u_{xx} + u_{yy} = 0} \Rightarrow u \text{ harmonic.}$$

We find the harmonic conjugate v . We have $v_x = -u_y = -(e^y - e^{-y}) \cos x - (e^y + e^{-y}) \sin x$

$$\Rightarrow v = -(e^y - e^{-y}) \sin x + (e^y + e^{-y}) \cos x + \phi(y). \text{ Since } v_y = u_x \Leftrightarrow$$

$$\Leftrightarrow -(e^y + e^{-y}) \sin x + (e^y - e^{-y}) \cos x + \phi'(y) = -(e^y + e^{-y}) \sin x + (e^y - e^{-y}) \cos x$$

$$\Rightarrow \phi'(y) = 0 \Rightarrow \phi = c, c \in \mathbb{R}. \text{ Since } f(0) = 2i \Rightarrow v(0) = 0 \Rightarrow c + 2 = 0 \Rightarrow \boxed{c = -2}$$

But ~~the~~ $\boxed{f(z) = u + iv} = \frac{(e^y + e^{-y}) \cos x + (e^y - e^{-y}) \sin x + i(e^y + e^{-y}) \cos x - i(e^y - e^{-y}) \sin x - 2i}{2} = (1+i)(e^y + e^{-y}) \cos x + (1-i)(e^y - e^{-y}) \sin x - 2i =$

$$= (1+i) \left[(e^y + e^{-y}) \cos x - i(e^y - e^{-y}) \sin x \right] - 2i =$$

$$= (1+i) \left[e^y (\cos x - i \sin x) + e^{-y} (\cos x + i \sin x) \right] - 2i =$$

$$= (1+i) (e^{y-ix} + e^{-y+ix}) - 2i = (1+i) (e^{-12} + e^{12}) - 2i = \boxed{2(1+i) \cos 2 - 2i}$$

4] Since u, v are harmonic conjugates $\Rightarrow f = u + iv$ is holomorphic. We need to check

$F = u(x^2 - y^2, 2xy) + i v(x^2 - y^2, 2xy)$ is holomorphic. But $z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$ so

$F = u(z^2) + i v(z^2) = f(z^2)$ which is clearly holomorphic.

5] Clearly, if $z=0$, $f(0) = 0$ so $|f(z)|$ is minimized at $z=0$. (and $\forall z \in \mathbb{R}, f(z) \neq 0$ so this min is unique).

For the max, we know it must occur on the boundary. We have to check the