

8 (i)  $\text{Res}_{z=\pi} \frac{z - \sin z}{z^2 \sin z} = \frac{f'(z)}{g'(z)}$  with  $f(z) = \frac{z - \sin z}{z^2}$ ,  $f(\pi) = \frac{1}{\pi}$ ,  $g = \sin z$ ,  $f(\pi) = \frac{1}{\pi}$ ,  $g'(\pi) = \cos \pi = -1$

Answer  $\boxed{-\frac{1}{\pi}}$

(ii).  $\text{Res}_{z=-i} \frac{\sqrt{z}}{(z^2+1)^2} = \text{Res}_{z=-i} \frac{f(z)}{(z+i)^2} = f'(-i)$  with  $f(z) = \frac{\sqrt{z}}{(z-i)^2}$ . Now

$f'(z) = \frac{1}{(z-i)^4} \left\{ \frac{1}{2} \cdot z^{-1/2} (z-i)^2 - z^{1/2} \cdot 2(z-i) \right\}$  For  $z=-i$ ,  $z^{1/2} = \left( e^{-\pi/2} \right)^{1/2} = e^{-\pi/4} = \frac{1-i}{\sqrt{2}}$

$z^{-1/2} = e^{\pi/4} = \frac{1+i}{\sqrt{2}} \Rightarrow f'(-i) = \frac{1}{(-2i)^4} \left( \frac{1}{2} \cdot \frac{1+i}{\sqrt{2}} (-2i)^2 - \frac{1-i}{\sqrt{2}} \cdot 2(-2i) \right) = \frac{1}{16} \left( -\sqrt{2}(1+i) + 2\sqrt{2} \frac{i(1-i)}{i+1} \right) = \boxed{\frac{1+i}{16} \cdot \sqrt{2}}$

9  $f(z) = \frac{1}{z+1} - \frac{1}{z+4}$ . When  $|z| < 1$ ,  $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$   
 When  $|z| > 1$ ,  $\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{\frac{1}{z}+1} = +\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \dots$

When  $|z| < 4$ ,  $\frac{1}{z+4} = \frac{1}{4} \cdot \frac{1}{\frac{z}{4}+1} = \frac{1}{4} \left( 1 - \frac{z}{4} + \frac{z^2}{4^2} - \dots \right)$   
 When  $|z| > 4$ ,  $\frac{1}{z+4} = \frac{1}{z} \cdot \frac{1}{1+\frac{4}{z}} = \frac{1}{z} \left( 1 - \frac{4}{z} + \frac{4^2}{z^2} - \dots \right)$

When  $|z| < 1$ ,  $f(z) = \left( 1 - \frac{1}{4} \right) - z \left( 1 - \frac{1}{4^2} \right) + z^2 \left( 1 - \frac{1}{4^3} \right) - \dots$

$1 < |z| < 4$ ,  $f(z) = \left( +\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \dots \right) - \left( \frac{1}{4} - \frac{z}{4^2} + \frac{z^2}{4^3} - \dots \right)$

$|z| > 4$ ,  $f(z) = \frac{3}{z^2} + \frac{1-4^2}{z^3} + \frac{-1+4^3}{z^4} + \dots$

10 (i) isolated singularities at  $0, a, b$ , all are poles.

$\text{Res}_{z=0} f(z) = \frac{1}{ab}$ ,  $\text{Res}_{z=a} f(z) = \frac{1}{z(z-b)} \Big|_{z=a} = \frac{1}{a(a-b)}$ ,  $\text{Res}_{z=b} f(z) = \frac{1}{z(z-a)} \Big|_{z=b} = \frac{1}{b(b-a)}$

(ii)  $\int_C f(z) dz = 2\pi i \left( \frac{1}{ab} + \frac{1}{a(a-b)} + \frac{1}{b(b-a)} \right) = 2\pi i \cdot \frac{(a-b) + b - a}{ab(a-b)} = \boxed{0}$

(iii)  $\frac{1}{z-a} = \frac{1}{z \left( 1 - \frac{a}{z} \right)} = \frac{1}{z} \left( 1 + \frac{a}{z} + \dots \right)$   
 $\frac{1}{z-b} = \frac{1}{z \left( 1 - \frac{b}{z} \right)} = \frac{1}{z} \left( 1 + \frac{b}{z} + \dots \right)$   
 $\Rightarrow f(z) = \frac{1}{z^3} \left( 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots \right) \left( 1 + \frac{b}{z} + \frac{b^2}{z^2} + \dots \right)$