Math 10C - Winter 2010 - Final Exam

Name: ________________________________

Student ID: __________________________

Section time: _________________________

Instructions:

During the test, you may not use books or telephones. You may use a "cheat sheet" of notes which should be a page.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 10 questions which are worth 120 points. You have 180 minutes to complete the test.

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Problem 1. [13 points.]

Consider the function
\[ f(x, y) = x^2 y - x \ln y. \]

(i) [4] Find the equation of the tangent plane to the graph of \( f \) at \( (2, 1, 4) \).

(ii) [3] Find a vector normal to the graph of \( f \).
(iii) [6] Find the quadratic Taylor polynomial of $f$ near the point $(2, 1)$. 
Problem 2. [8 points.]

Calculate the sum
\[
\frac{2}{3^3} + \frac{2^2}{3^4} + \frac{2^3}{3^5} + \ldots + \frac{2^{2010}}{3^{2012}}.
\]
Please express your answer in simplest form.
Problem 3. [12 points.]

Consider the function

\[ f(x, y) = x^2 y - xy^2 - 3x. \]

(i) [7] Find the critical points of the function.
(ii) [5] Classify the critical points as local max/local min/saddle.
Problem 4. [15 points.]

Consider the function
\[ f(x, y) = (x - y)^2 \sqrt{x + 2y}. \]

(i) [5] Calculate the gradient of \( f \) at the point \((2, 1)\).

(ii) [4] Calculate the tangent line to the level curve of \( f \) passing through the point \((2, 1)\).
(iii) [3] Calculate the unit direction of steepest increase for the function $f$ at $(2, 1)$.

(iv) [3] Calculate the rate of change of the function $f$ at the point $(2, 1)$ in the direction

$$\vec{u} = \frac{-3\vec{i} - 4\vec{j}}{5}.$$
Problem 5. [12 points.]

Find the maximal value of the function

\[ f(x, y) = (x + 1)^2 + (y - 2)^2 \]

over the compact region

\[ x^2 + y^2 \leq 20. \]
Problem 6. [12 points.]

Consider the points \( P(1, 1, -1), Q(1, 2, 0) \) and \( R(2, 1, 2) \).

(i) [5] Find the equation of the plane through \( P, Q \) and \( R \).

(ii) [3] Find the area of the triangle \( PQR \).

(iii) [4] Find the angle between the vectors \( \vec{PQ} \) and \( \vec{PR} \).
Problem 7. [12 points.]

Find the maximum value of the expression

\[ f(x, y, z) = x + 3y + 4z \]

subject to the constraint

\[ x^2 + y^2 + z^2 = 26. \]
Consider the function

\[ f(x, y) = \frac{y - 1}{x^2}. \]

(i) [4] Where is the function defined? Draw the contour diagram of \( f \) showing at least three different levels.

(ii) [3] Draw the cross-section \( x = 1 \) of the graph of \( f \). Describe the cross-section in words.
(ii) [3] Does the function have a global maximum? If so, where does it occur?

(iii) [3] Does the function have a global maximum over the rectangle

$$1 \leq x \leq 2, 1 \leq y \leq 2.$$ 

If so, where does it occur?
Problem 9. [10 points.]

Assume that
\[ w = e^{-xy^2} \]
and that
\[ x = s^2 t, \quad y = \frac{1}{t}. \]

Calculate the partial derivatives
\[ \frac{\partial w}{\partial s} \text{ and } \frac{\partial w}{\partial t}. \]
Problem 10. [13 points.]

The cumulative distribution function for the outcome of a certain experiment is

\[ P(x) = \begin{cases} 
1 - \frac{1}{x^3} & \text{if } x \geq 1 \\
0 & \text{otherwise} 
\end{cases} \]

(i) [3] Find the probability density function.

(ii) [3] What percentage of outcomes have values \( x \geq 2 \)?
(iii) [3] Find the median of the experiment.

(iv) [4] What is the mean value of the experiment?