Problem 1. [15 points]
At what point \((x, y, z)\) on the plane \(x + 2y - z = 5\) does the minimum of the function
\[
f(x, y, z) = x^2 + 2y^2 + (z + 1)^2
\]
occur?

Answer:
Using Lagrange multipliers, we need to solve
\[
\nabla f = \lambda \nabla g
\]
where
\[
g(x, y, z) = x + 2y - z.
\]
Computing the gradients, we conclude
\[
(2x, 4y, 2(z + 1)) = \lambda(1, 2, -1) \implies x = \frac{\lambda}{2}, y = \frac{\lambda}{2}, z = -\frac{\lambda}{2} - 1.
\]
Substituting, we have
\[
x + 2y - z = 5 \implies \frac{\lambda}{2} + 2 \cdot \frac{\lambda}{2} + \frac{\lambda}{2} + 1 = 5 \implies \lambda = 2.
\]
This gives
\[
x = 1, y = 1, z = -2.
\]

Problem 2. [20 points.]
Consider the function
\[
f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy.
\]
(i) [8] Find the critical points of the function.
(ii) [8] Determine the nature of the critical points (local min/local max/saddle).
(iii) [4] Does the function \(f(x, y)\) have a global minimum or a global maximum?

Answer:
(i) We have
\[
f_x = -6x + 6y = 0 \implies x = y
\]
\[
f_y = 6y - 6y^2 + 6x = 0 \implies y - y^2 + x = 0.
\]
Substituting \(x = y\) into the second equation, we obtain
\[
2y - y^2 = 0 \implies y = 0 \text{ or } y = 2.
\]
The critical points are
\[
(0, 0), (2, 2).
\]
(ii) We compute
\[
A = f_{xx} = -6, \quad B = f_{xy} = 6, \quad f_{yy} = 6 - 12y.
\]
When
\[
\begin{align*}
x = y = 0 & \implies AC - B^2 = (-6)(6) - 6^2 < 0 \implies (0, 0) \text{ saddle point,} \\
x = y = 2 & \implies AC - B^2 = (-6)(-18) - 6^2 > 0, A < 0 \implies (2, 2) \text{ local max.}
\end{align*}
\]
(iii) We compute
\[
\lim_{{x \to \infty, y \to 0}} f(x, y) = \lim_{{x \to \infty}} -3x^2 = -\infty \implies \text{no global min.}
\]
Similarly,
\[
\lim_{{x=0, y \to -\infty}} f(x, y) = \lim_{{y \to -\infty}} 3y^2 - 2y^3 = \infty \implies \text{no global max.}
\]

**Problem 3. [13 points]** Consider the function
\[
f(x, y) = \ln(xy^2) - \frac{2x}{y}
\]
(i) [8] Compute the second order Taylor polynomial of \(f\) around \((1, 1)\).
(ii) [5] Find the tangent plane to the graph of \(f\) at the point \((1, 1, -2)\).

**Answer:**
(i) We compute
\[
\begin{align*}
f(1, 1) &= \ln 1 - 2 = -2 \\
f_x(x, y) &= y^2 - \frac{2}{y} = \frac{1}{x} - \frac{2}{y} \implies f_x(1, 1) = -1 \\
f_y(x, y) &= 2xy - \frac{2x}{y^2} = \frac{2}{y} + \frac{2x}{y^2} \implies f_y(1, 1) = 4 \\
f_{xx} &= -\frac{1}{x^2} \implies f_{xx}(-1, 1) = -1 \\
f_{xy} &= \frac{2}{y^2} \implies f_{xy}(1, 1) = 2 \\
f_{yy} &= -\frac{2}{y^2} - \frac{4x}{y^3} \implies f_{yy}(1, 1) = -6.
\end{align*}
\]
The Taylor polynomial is
\[
-2 - (x - 1) + 4(y - 1) - \frac{1}{2}(x - 1)^2 + 2(x - 1)(y - 1) - 3(y - 1)^2.
\]
(ii) The tangent plane is the linear part of the Taylor polynomial
\[
z = -2 - (x - 1) + 4(y - 1) = -x + 4y - 5.
\]

**Problem 4. [17 points.]**

Find the global minimum and global maximum of the function
\[
f(x, y) = x^2 + y^2 - 2x - 2y + 4
\]
over the closed disk
\[
x^2 + y^2 \leq 8.
\]

**Answer:**
We find the critical points in the interior by setting the partial derivatives to zero:
\[
\begin{align*}
f_x &= 2x - 2 = 0 \implies x = 1 \\
f_y &= 2y - 2 = 0 \implies y = 1.
\end{align*}
\]
We get the critical point \((1, 1)\) with value
\[ f(1, 1) = 2. \]
We check the boundary \(g(x, y) = x^2 + y^2 = 8\) using Lagrange multipliers
\[ \nabla f = \lambda \nabla g \implies (2x - 2, 2y - 2) = \lambda (2x, 2y) \implies x - 1 = \lambda x, y - 1 = \lambda y. \]
Dividing we obtain
\[ \frac{x - 1}{y - 1} = \frac{\lambda x}{\lambda y} \implies y(x - 1) = x(y - 1) \implies xy - y = xy - x \implies x = y. \]
Since
\[ x^2 + y^2 = 8 \implies x = y = \pm 2. \]
We evaluate
\[ f(2, 2) = 4, \quad f(-2, -2) = 20. \]
Therefore \((1, 1)\) is the global minimum, while \((-2, -2)\) is the global maximum.

**Problem 5.** [15 points]
Consider the function
\[ f(x, y) = 1 + x^2 + y^2. \]
(i) [4] Draw the contour diagram of \(f\) labeling at least three levels of your choice.
(ii) [4] Compute the gradient of \(f\) at \((1, -1)\) and draw it on the contour diagram of part (i).
(iii) [3] Does the function \(f\) have a global minimum? If no, why not? If yes, what is the minimum value?
(iv) [4] Draw the graph of the function \(f\).

**Answer:**
(i) The contour diagram consists of concentric circles of centered at the origin:
\[ f(x, y) = c \implies x^2 + y^2 = c - 1. \]
Level \(c\) corresponds to a circle of radius \(\sqrt{c - 1}\).
You may use three values for \(c\) to draw the contour diagram. For instance, for \(c = 2, 5, 10\), we get circles of radii 1, 2, 3 respectively.
(ii) The gradient is
\[ \nabla f = (2x, 2y) \implies \nabla f(1, -1) = (2, -2). \]
This vector is normal to the level curve.
(iii) The global minimum occurs at \((0, 0)\). The minimum value is \(f(0, 0) = 1\).
(iv) The graph is a paraboloid with lowest point at \((0, 0, 1)\).

**Problem 6.** [15 points]
Consider the function
\[ f(x, y) = e^{-3x + 2y} \sqrt{2x + 1}. \]
(i) [5] Calculate the gradient of \(f\) at \((0, 0)\).
(ii) [5] Find the directional derivative of \(f\) at \((0, 0)\) in the direction \(u = \frac{i + j}{\sqrt{2}}\).
(iii) [5] What is the unit direction for which the rate of increase of \(f\) at \((0, 0)\) is maximal?
Answer:

(i) Using the product rule and the chain rule, we have
\[ f_x = -3e^{-3x+2y}\sqrt{2x+1} + e^{-3x+2y} \cdot \frac{1}{2} \sqrt{2x+1} \cdot 2 \implies f_x(0,0) = -3 + 1 = -2, \]
\[ f_y = 2e^{-3x+2y}\sqrt{2x+1} \implies f_y(0,0) = 2. \]
The gradient is \( \nabla f = (-2, 2) \).

(ii) We compute
\[ f_u = \nabla f \cdot u = (-2, 2) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 0. \]

(iii) The direction is parallel to the gradient, normalized to have unit length
\[ v = \frac{\nabla f}{\|\nabla f\|} = \frac{(-2, 2)}{\sqrt{(-2)^2 + 2^2}} = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \]

Problem 7. [15 points]
Consider the planes
\[ x + 2y - z = 1, \ x + 4y - 2z = 3. \]

(i) [4] Find normal vectors to the two planes.
(ii) [6] Are the two planes parallel? Are they perpendicular?
(iii) [5] Find a vector parallel to the line of intersection of the two planes.

Answer:

(i) The normal vectors are given by the coefficients of the two planes
\[ \vec{n}_1 = (1, 2, -1), \ \vec{n}_2 = (1, 4, -2). \]

(ii) The vectors \( \vec{n}_1, \vec{n}_2 \) are not proportional hence the planes are not parallel. We have
\[ \vec{n}_1 \cdot \vec{n}_2 = (1, 2, -1) \cdot (1, 4, -2) = 1 + 8 - 3 \neq 0. \]
The planes are not perpendicular.

(iii) The line of intersection is normal to both \( n_1 \) and \( n_2 \) hence it is parallel to the cross product
\[ \vec{n}_1 \times \vec{n}_2 = (0, 1, 2). \]

Problem 8. [15 points.]
Consider the function
\[ w = u^2 v e^{-v} \]
and assume that
\[ u = x^2 - 2xy, \ v = -x + 2 \ln y. \]
Calculate the values of the derivatives
\[ \frac{\partial w}{\partial x} \text{ and } \frac{\partial w}{\partial y} \]
at the point \((x, y) = (1, 1)\).
Answer: We evaluate
\[ \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}. \]
At \( x = y = 1 \), we have \( u = -1, v = -1 \).

We compute

\[
\frac{\partial w}{\partial u} = 2uve^{-v} = 2e
\]

\[
\frac{\partial w}{\partial v} = u^2e^{-v} - u^2ve^{-v} = e + e = 2e.
\]

Furthermore,

\[
\frac{\partial u}{\partial x} = 2x - 2y = 0
\]

\[
\frac{\partial v}{\partial x} = -1.
\]

Thus

\[
\frac{\partial w}{\partial x} = 2e \cdot 0 + 2e \cdot (-1) = -2e.
\]

A similar calculation shows

\[
\frac{\partial w}{\partial y} = 0.
\]

**Problem 9. [10 points]**

You deposit $1,000 into your savings account every year for the next 10 years. You make the first deposit on January 1, 2010, and the last deposit on January 1, 2019. The interest rate for the account is \( r = 10\% \) per year. How much money will there be in your account at the end of the 10th year, on December 31, 2019?

Express your answer in the simplest closed form. You don’t need to evaluate the powers that may appear in the final expression.

**Answer:**

After 1 year, on January 1, 2011 you have 1,000 from new deposits and 1.1 \cdot 1,000 from deposits and interest over the previous year, a total of

\[
1,000 + 1.1 \cdot 1,000.
\]

On January 1, 2012, you own

\[
1.1 \cdot (1,000 + 1.1 \cdot 1,000)
\]

from interest and 1,000 from new deposits, a total of

\[
1,000 + 1.1 \cdot (1,000 + 1.1 \cdot 1,000) = 1,000 + 1.1 \cdot 1,000 + (1.1)^2 \cdot 1,000.
\]

On January 1, 2013, you own

\[
1,000 + 1.1 \cdot (1,000 + 1.1 \cdot 1,000 + (1.1)^2 \cdot 1,000) = 1,000 + 1,000 \cdot 1.1 + 1,000 \cdot (1.1)^2 + 1,000 \cdot (1.1)^3.
\]

Continuing in this fashion, on January 1, 2019 you own

\[
1,000 + 1,000 \cdot 1.1 + \ldots + 1,000 \cdot (1.1)^9.
\]

On December 31, 2019 you collect interest hence the total sum is

\[
1.1 \cdot (1,000 + 1,000 \cdot 1.1 + \ldots + 1,000 \cdot (1.1)^9).
\]

This is a geometric series with step 1.1, hence the sum equals

\[
1.1 \cdot 1,000 \cdot \frac{1 - (1.1)^{10}}{1 - 1.1} = 1.1 \cdot 1,000 \cdot \frac{(1.1)^{10} - 1}{-0.1} = 11,000 \cdot ((1.1)^{10} - 1).
\]
Problem 10. [15 points]

The outcome $x$ of a certain experiment has values between $0$ and $\frac{\pi}{2}$, with probability distribution function

$$p(x) = \sin x, \text{ for } 0 \leq x \leq \frac{\pi}{2}.$$ 

(i) [4] Calculate the cumulative distribution function.

(ii) [4] Calculate the median outcome of the experiment.

(iii) [7] Possibly using integration by parts, calculate the mean outcome of the experiment.

Answer:

(i) To find the cdf, we integrate the pdf:

$$P(t) = \int_0^t p(x) \, dx = \int_0^t \sin x \, dx = -\cos x \bigg|_{x=0}^{x=t} = -\cos t + 1.$$ 

(ii) The median is found by solving

$$P(T) = \frac{1}{2} \implies 1 - \cos T = \frac{1}{2} \implies \cos T = \frac{1}{2} \implies T = \frac{\pi}{3}.$$ 

(iii) The mean is computed by the integral which can be evaluated by parts

$$\mu = \int_0^{\frac{\pi}{2}} x \sin x \, dx = -x \cos x \bigg|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx = 0 + \sin x \bigg|_0^{\frac{\pi}{2}} = 1.$$