Math 10C - Winter 2010 - Final Exam

Name: __________________________________________

Student ID: _________________________________

Section time: _______________________________

Instructions:

During the test, you may not use books or telephones. You may use a "cheat sheet" of notes which should be a page.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 10 questions which are worth 120 points. You have 180 minutes to complete the test.

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Problem 1. [13 points.]

Consider the function

\[ f(x, y) = x^2 y - x \ln y. \]

(i) [4] Find the equation of the tangent plane to the graph of \( f \) at \((2, 1, 4)\).

(ii) [3] Find a vector normal to the graph of \( f \).
(iii) [6] Find the quadratic Taylor polynomial of $f$ near the point $(2,1)$. 
Problem 2. [8 points.]

Calculate the integral

$$\int_R xy \, dx \, dy$$

where $R$ is the triangular region $x \geq 0, y \geq 0, x + 2y \leq 2$. 
Problem 3. [12 points.]

Find the critical points of the function

\[
f(x, y) = xy^2 - 4xy + \frac{1}{2}x^2
\]

and determine their nature.
Problem 4. [15 points.]

Consider the function

\[ f(x, y) = (x - y)^2 \sqrt{x + 2y}. \]

(i) [5] Calculate the gradient of \( f \) at the point \((1, 0)\).

(ii) [5] Calculate the tangent line to the level curve of \( f \) passing through the point \((1, 0)\).
(iii) [5] Calculate the unit direction of steepest increase for the function $f$ at $(1,0)$. 
Problem 5. [12 points.]

Find the maximal value of the function

\[ f(x, y) = (x + 1)^2 + (y - 2)^2 \]

over the compact region

\[ x^2 + y^2 \leq 20. \]
Problem 6. [12 points.]

Consider the points $P(1,1,-2)$, $Q(2,0,1)$ and $R(1,-1,0)$.

(i) [5] Find the area of the triangle $PQR$.

(ii) [4] Find the equation of the plane through $P$, $Q$ and $R$.

(ii) [4] Find the cosine of the angle between the vectors $\vec{PQ}$ and $\vec{QR}$. 
Problem 7. [12 points.]

Find the minimum and the maxim value of the function

\[ f(x, y, z) = 2x - 2y + z \]

along the sphere of center \((1, 0, -1)\) and radius 3.
Problem 8. [13 points.]

Consider the function \( f(x, y) = \frac{x^2}{y^4} \).

(i) [4] Where is the function defined? Carefully draw the level curve passing through \((1, -1)\).

On this graph, draw the gradient of the function at \((1, -1)\).

(ii) [3] Compute the directional derivative of \( f \) at \((1, -1)\) in the direction \( \mathbf{u} = \left( \frac{4}{5}, \frac{3}{5} \right) \). Use this calculation to estimate

\[ f((1, -1) + .01\mathbf{u}) \].
(iii) [3] Does the function $f$ have a global maximum?

(iv) [3] Does the function $f$ have a maximum over the region $1 \leq x \leq 2, 1 \leq y \leq 2$. If so, where does it occur?
Problem 9. [10 points.]

Consider the function

\[ w = e^{x^2y} \]

where

\[ x = u\sqrt{v}, \quad y = \frac{1}{uv^2}. \]

Using the chain rule, compute the derivatives

\[ \frac{\partial w}{\partial u}, \quad \frac{\partial w}{\partial v}. \]
Problem 10. [13 points.]

Change the order of integration in the following integral and evaluate the answer explicitly
\[ \int_0^2 \int_0^{4-y^2} x \, dx \, dy. \]