Problem 1. [10 points.]

Consider the vectors

\[ v = \mathbf{i} + 2\mathbf{j}, \quad w = 3\mathbf{i} - \mathbf{j}. \]

(i) Draw the two vectors \( v \) and \( w \) in standard position, and draw their sum \( v + w \).
(ii) Compute the vector \( 2v + 3w \).
(iii) Find the unit vector \( u \) in the direction of \( w \).

Answer: (i) The vector \( v + w \) in standard position joins the origin to the vertex of the parallelogram spanned by \( v \) and \( w \). Picture omitted.
(ii) We have

\[ 2v + 3w = 2(\mathbf{i} + 2\mathbf{j}) + 3(3\mathbf{i} - \mathbf{j}) = 11\mathbf{i} + \mathbf{j}. \]

(iii) We have

\[ u = \frac{w}{||w||} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{3^2 + (-1)^2}} = \frac{3}{\sqrt{10}} \mathbf{i} - \frac{1}{\sqrt{10}} \mathbf{j}. \]

\( \square \)

Problem 2. [10 points.]

Consider the function

\[ f(x, y) = (x - 1)^2 + (y - 1)^2. \]

(i) Draw the contour diagram for \( f(x, y) \) and clearly label the level curves. Show the contours for at least three levels.
(ii) Draw the graph of \( z = f(x, y) \).

Answer: (i) We set \( f(x, y) = c \) which gives \((x - 1)^2 + (y - 1)^2 = c\). The contour diagram consists in circles centered at \((1, 1)\) of radius \( \sqrt{c} \) where \( c \) is the level. For instance, for levels \( c = 1, c = 4, c = 9 \), we draw three circles of center \((1, 1)\) and radii \(1, 2\) and \(3\) respectively.
(ii) The graph \( z = f(x, y) \) is a paraboloid whose lowest point is \((1, 1, 0)\).

\( \square \)

Problem 3. [10 points.]

(i) Suppose that \( z = f(x, y) \) is a linear function of \( x \) and \( y \), with slope 3 in the \( x \) direction and slope \(-2\) in the \( y \) direction. A change of \(1\) in \( x \) and \(-.2\) in \( y \) produces what change in \( z \)?
(ii) The graph of a linear function \( z = g(x, y) \) passes through the point \((1, 2, 5)\). The graph intersects the \( xz \) plane along the line \( z = 3x + 6 \). Determine the linear function \( g \).

Answer: (i) We know that \( m = 3 \) and \( n = -2 \). Thus

\[ \Delta z = 3\Delta x - 2\Delta y. \]

In our case \( \Delta x = .1 \) and \( \Delta y = -.2 \). The change in \( z \) is therefore

\[ \Delta z = 3(.1) - 2(-.2) = .7. \]

Answer: (ii) We know that \( m = 3 \) and \( n = -2 \). Thus

\[ \Delta z = 3\Delta x - 2\Delta y. \]

In our case \( \Delta x = .1 \) and \( \Delta y = -.2 \). The change in \( z \) is therefore

\[ \Delta z = 3(.1) - 2(-.2) = .7. \]
(ii) The graph of the linear function is
\[ z = c + mx + ny. \]
The intersection with the \( xz \) plane is given by setting \( y = 0 \). The equation we obtain is \( z = c + mx \) which we know should be \( z = 3x + 6 \). We obtain
\[ c = 6, m = 3. \]
Now, since \((1, 2, 5)\) lies on the graph we have
\[ 5 = c + m + 2n. \]
We determine \( n = -2 \). The linear function is
\[ f(x, y) = 6 + 3x - 2y. \]

Problem 4. [10 points.]
Consider the points \( P(1, 1, -1), Q(1, 2, 0) \) and \( R(2, 1, 2) \).

(i) Find the equation of the plane through \( P, Q \) and \( R \).

(ii) Find the cosine of the angle between the vectors \( \vec{PQ} \) and \( \vec{PR} \).

(iii) Find the area of the triangle \( PQR \).

Answer: (i) We have
\[ \vec{PQ} = (0, 1, 1) \) and \( \vec{PR} = (1, 0, 3). \)
(These vectors were found by taking difference in coordinates.) We find a normal vector to the plane by computing the cross product. Using the rules of computing cross product we find
\[ \vec{PQ} \times \vec{PR} = 3i + j - k. \]
The equation of the plane can be found from the normal vector \((3, -1, -1)\) and using the reference point \( P(1, 1, -1) \) to be
\[ 3(x - 1) + (y - 1) - (z + 1) = 0 \text{ or } 3x + y - z = 5. \]

(ii) To find the angle, we compute the dot product
\[ \vec{PQ} \cdot \vec{PR} = (0, 1, 1) \cdot (1, 0, 3) = 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 3 = 3. \]
We also find
\[ ||\vec{PQ}|| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \]
\[ ||\vec{PR}|| = \sqrt{1^2 + 0^2 + 3^2} = \sqrt{10}. \]
Thus
\[ \cos \theta = \frac{\vec{PQ} \cdot \vec{PR}}{||\vec{PQ}|| \cdot ||\vec{PR}||} = \frac{3}{\sqrt{2} \cdot \sqrt{10}} = \frac{3}{\sqrt{20}}. \]

(iii) The area of the triangle \( PQR \) is half the area of the parallelogram spanned by the vectors \( \vec{PQ} \) and \( \vec{PR} \) which equals
\[ ||\vec{PQ} \times \vec{PR}|| = ||3i + j - k|| = \sqrt{3^2 + 1^2 + (1)^2} = \sqrt{11}. \]
Thus the area of the triangle is \( \frac{1}{2} \sqrt{11}. \)