1. Find the critical points of the function \( f(x, y) = \frac{1}{2}x^2 + \frac{3}{2}y^2 - xy^3 \) and indicate their type.

Answer: We have
\[
f_x = x - y^3 = 0 \implies x = y^3
\]
\[
f_y = 3y - 3xy^2 = 0 \implies y = xy^2 \implies y = y^5 \implies y = 0, y = 1, \text{ or } y = -1.
\]
We find the critical points \((0, 0), (1, 1), (-1, -1)\).

We calculate
\[
f_{xx} = 1, \ f_{xy} = -3y^2, \ f_{yy} = 3 - 6xy.
\]
At the critical point \((0, 0)\) the Hessian is
\[
H_f = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}
\]
which is positive definite, hence \((0, 0)\) is a local minimum. At \((x, y) = \pm (1, 1)\), the Hessian is
\[
H_f = \begin{bmatrix} 1 & -3 \\ -3 & -3 \end{bmatrix}
\]
which has negative determinant, hence it is definite, so \( \pm (1, 1) \) are both saddle points.

2. Find the second order Taylor polynomial near \((1, -1)\) for the function
\[
f(x, y) = x^3y.
\]

Answer: We compute
\[
f(1, -1) = -1,
\]
\[
f_x = 3x^2y \implies f_x(1, -1) = -3
\]
\[
f_y = x^3 \implies f_y(1, -1) = 1,
\]
\[
f_{xx} = 6xy \implies f_{xx}(1, -1) = -6,
\]
\[
f_{xy} = 3x^2 \implies f_{xy}(1, -1) = 3,
\]
\[
f_{yy} = 0.
\]
The Taylor polynomial is
\[
P_2 = -1 - 3(x - 1) + (y + 1) - 3(x - 1)^2 + 3(x - 1)(y + 1).
\]

3. Consider the function
\[
f(x, y) = x^4y^3.
\]

(i) Write down the equation of the tangent plane at the graph of the function at the point \((1, 1, 1)\).
(ii) Write down an expression for the change, \(\Delta z\), in \(z = f(x, y)\) depending on \(\Delta x\) and \(\Delta y\), the change in \(x\) and \(y\), respectively, near the point \(x = y = 1\). Is the function \(f(x, y)\) more sensitive to a change in \(x\) or to a change in \(y\)?

(iii) Using your answer to (ii), find the approximate value of \(f(1.01, 1.02)\).

**Answer:**

(i) We compute

\[
\begin{align*}
f_x &= 4x^3 y^3 \implies f_x(1, 1) = 4, \\
f_y &= 3x^4 y^2 \implies f_y(1, 1) = 3.
\end{align*}
\]

The tangent plane is

\[
z - 1 = 4(x - 1) + 3(y - 1) \implies 4x + 3y - z = 6.
\]

(ii)

\[
\Delta z = 4\Delta x + 3\Delta y.
\]

The function is more sensitive to a change in \(x\) because the \(x\) derivative at \((1, 1)\) is higher.

(iii) We have

\[
\Delta x = 1.01 - 1 = .01, \Delta y = 1.02 - 1 = .02,
\]

hence

\[
\Delta z = 4(.01) + 3(.02) = .1.
\]

This gives

\[
z(1.01, 1.02) = z(1, 1) + \Delta z = 1.1 \implies f(1.01, 1.02) \approx 1.1.
\]

\[
\square
\]

4. Consider the function \(f(x, y) = xe^{x+y}\) and the point \(P = (2, -2)\).

(i) Find the gradient of \(f\) at \(P\).

(ii) Find the directional derivative of \(f\) at \(P\) in the direction \(\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})\).

(iii) What is the direction of steepest increase for the function \(f\) at \(P\)? Express your answer as a unit vector.

**Answer:**

(i)

\[
\begin{align*}
f_x &= e^{x+y} + xe^{x+y} \implies f_x(2, -2) = e^0 + 2e^0 = 3, \\
f_y &= xe^{x+y} \implies f_y(2, -2) = 2e^0 = 2.
\end{align*}
\]

We have

\[
\nabla f(P) = (3, 2).
\]

(ii)

\[
f_\mathbf{u}(P) = \nabla f(P) \cdot \mathbf{u} = (3, 2) \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}.
\]

(iii) The direction of steepest increase is given by the gradient. Since we want a unit vector, we divide by the length

\[
\mathbf{v} = \frac{(3, 2)}{||(3, 2)||} = \frac{(3, 2)}{\sqrt{13}}.
\]

\[
\square
\]
5. Consider the function \( w = \sin(xy) \)

where \( x = \frac{1}{v}, \quad y = u^2 v. \)

Using the chain rule, calculate the derivatives

\( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v}. \)

Please express your answer in simplest form.

Answer: We compute

\[
\frac{\partial w}{\partial x} = y \cos(xy) = u^2 v \cos(u^2), \quad \frac{\partial w}{\partial y} = x \cos(xy) = \frac{1}{v} \cos(u^2)
\]

\[
\frac{\partial x}{\partial u} = 0, \quad \frac{\partial x}{\partial v} = -\frac{1}{v^2}, \quad \frac{\partial y}{\partial u} = 2uv, \quad \frac{\partial y}{\partial v} = u^2.
\]

Then

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} = u^2 v \cos(u^2) \cdot 0 + \frac{1}{v} \cos(u^2) \cdot (2uv) = 2u \cos(u^2)
\]

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = u^2 v \cos(u^2) \cdot -\frac{1}{v^2} + \frac{1}{v} \cos(u^2) \cdot u^2 = 0.
\]

\( \square \)