1. Find the critical points of the function \( f(x, y) = \frac{1}{2}x^2 + \frac{3}{2}y^2 - xy^3 \) and indicate their type.

Answer: We have
\[
\begin{align*}
    f_x &= x - y^3 = 0 \implies x = y^3 \\
    f_y &= 3y - 3xy^2 = 0 \implies y = xy^2 \implies y = y^5 \implies y = 0, y = 1, \text{ or } y = -1.
\end{align*}
\]
We find the critical points \((0,0), (1,1), (-1, -1)\).

We calculate
\[
A = f_{xx} = 1, \ B = f_{xy} = -3y^2, \ C = f_{yy} = 3 - 6xy.
\]
At the critical point \((0,0)\) we have
\[
A = 1, \ B = 0, \ C = 3 \implies AC - B^2 > 0, \ A > 0
\]
so \((0,0)\) is a local minimum. At \((x, y) = \pm (1,1)\), we have
\[
A = 1, \ B = -3, \ C = -3 \implies AC - B^2 < 0
\]
so \pm (1,1) are both saddle points.

2. Find the second order Taylor polynomial near \((1, -1)\) for the function
\[
f(x, y) = x^3y.
\]

Answer: We compute
\[
\begin{align*}
    f(1, -1) &= -1, \\
    f_x &= 3x^2y \implies f_x(1, -1) = -3 \\
    f_y &= x^3 \implies f_y(1, -1) = 1, \\
    f_{xx} &= 6xy \implies f_{xx}(1, -1) = -6, \\
    f_{xy} &= 3x^2 \implies f_{xy}(1, -1) = 3, \\
    f_{yy} &= 0.
\end{align*}
\]
The Taylor polynomial is
\[
P_2 = -1 - 3(x - 1) + (y + 1) - 3(x - 1)^2 + 3(x - 1)(y + 1).
\]

3. Consider the function
\[
f(x, y) = x^4y^3.
\]
(i) Write down the equation of the tangent plane at the graph of the function at the point \((1,1,1)\).

(ii) Write down an expression for the change, \(\Delta z\), in \(z = f(x, y)\) depending on \(\Delta x\) and \(\Delta y\), the change in \(x\) and \(y\), respectively, near the point \(x = y = 1\). Is the function \(f(x, y)\) more sensitive to a change in \(x\) or to a change in \(y\)?

Using your answer to (ii), find the approximate value of \( f(1.01, 1.02) \).

**Answer:**

(i) We compute

\[
\begin{align*}
  f_x &= 4x^3 y^3 \implies f_x(1, 1) = 4 \\
  f_y &= 3x^4 y^2 \implies f_y(1, 1) = 3.
\end{align*}
\]

The tangent plane is

\[
  z - 1 = 4(x - 1) + 3(y - 1) \implies 4x + 3y - z = 6.
\]

(ii)

\[ \Delta z = 4\Delta x + 3\Delta y. \]

The function is more sensitive to a change in \( x \) because the \( x \) derivative at \((1, 1)\) is higher.

(iii) We have

\[ \Delta x = 1.01 - 1 = .01, \quad \Delta y = 1.02 - 1 = .02, \]

hence

\[ \Delta z = 4(.01) + 3(.02) = .1. \]

This gives

\[ z(1.01, 1.02) = z(1, 1) + \Delta z = 1.1 \implies f(1.01, 1.02) \approx 1.1. \]

\[ \Box \]

4. Consider the function \( f(x, y) = xe^{x+y} \) and the point \( P = (2, -2) \).

(i) Find the gradient of \( f \) at \( P \).

(ii) Find the directional derivative of \( f \) at \( P \) in the direction \( u = \frac{1}{\sqrt{2}}(i - j) \).

(iii) What is the direction of steepest increase for the function \( f \) at \( P \)? Express your answer as a unit vector.

**Answer:**

(i) \[
\begin{align*}
  f_x &= e^{x+y} + xe^{x+y} \implies f_x(2, -2) = e^0 + 2e^0 = 3, \\
  f_y &= xe^{x+y} \implies f_y(2, -2) = 2e^0 = 2.
\end{align*}
\]

We have

\[ \nabla f(P) = (3, 2). \]

(ii) \[
  f_u(P) = \nabla f(P) \cdot u = (3, 2) \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}.
\]

(iii) The direction of steepest increase is given by the gradient. Since we want a unit vector, we divide by the length

\[ v = \frac{(3, 2)}{|(3, 2)|} = \frac{(3, 2)}{\sqrt{13}}. \]

\[ \Box \]
5. Consider the function
\[ w = \sin(xy) \]
where
\[ x = \frac{1}{v}, \quad y = u^2v. \]
Using the chain rule, calculate the derivatives
\[ \frac{\partial w}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v}. \]
Please express your answer in simplest form.

Answer: We compute
\[
\begin{align*}
\frac{\partial w}{\partial x} &= y \cos(xy) = \frac{v}{u}v \cos(u^2), \quad \frac{\partial w}{\partial y} = x \cos(xy) = \frac{1}{v} \cos(u^2) \\
\frac{\partial x}{\partial u} &= 0, \quad \frac{\partial x}{\partial v} = -\frac{1}{v^2} \\
\frac{\partial y}{\partial u} &= 2uv, \quad \frac{\partial y}{\partial v} = u^2.
\end{align*}
\]
Then
\[
\begin{align*}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{v}{u}v \cos(u^2) \cdot 0 + \frac{1}{v} \cos(u^2) \cdot \frac{1}{v} = 2u \cos(u^2) \\
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{v}{u}v \cos(u^2) \cdot -\frac{1}{v^2} + \frac{1}{v} \cos(u^2) \cdot u^2 = 0.
\end{align*}
\]
\[ \square \]