Problem 1.

Consider the vectors

$$\vec{v} = \vec{i} + 2\vec{j}, \;\; \vec{w} = 3\vec{i} + \vec{j}.$$

(i) Compute the unit vector $\vec{u}$ in the direction of $\vec{v}$.
(ii) Find the component $\vec{w}_{||}$ of $\vec{w}$ in the direction parallel to $\vec{u}$.
(iii) Find the angle between the vectors $\vec{v}$ and $\vec{w}$.

`Answer:`

(i) *We have* $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$. *We compute*

$$||\vec{v}|| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

Thus

$$\vec{u} = \frac{1}{\sqrt{5}}\vec{v} = \frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}.$$

(ii) *We have*

$$w_{||} = (\vec{w} \cdot \vec{u}) \cdot \vec{u}.$$

We compute

$$\vec{w} \cdot \vec{u} = \frac{1}{\sqrt{5}}(1, 2) \cdot (3, 1) = \frac{3}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

Thus

$$w_{||} = \sqrt{5} \cdot u = \sqrt{5} \cdot \frac{1}{\sqrt{5}}(1, 2) = (1, 2) \;\; \textit{or} \; \vec{i} + 2\vec{j}.$$

(iii) *We have* $\vec{v} \cdot \vec{w} = (1, 2) \cdot (3, 1) = 5$. *Also* $||\vec{v}|| = \sqrt{5}$ *and* $||\vec{w}|| = \sqrt{3^2 + 1^2} = \sqrt{10}$. *Then*

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||} = \frac{5}{\sqrt{5} \cdot \sqrt{10}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}.$$

Problem 2.

Consider the function

$$f(x,y) = 1 + \sqrt{x^2+y^2}.$$

(i) Draw the contour diagram for $f(x,y)$ and clearly label the level curves. Show the contours for at least three levels.

(ii) Draw the graph of $z = f(x,y)$.

`Answer:`

(i) *The level curve at level c is*

$$f(x,y) = c \implies \sqrt{x^2+y^2} = c-1 \implies x^2+y^2 = (c-1)^2.$$

This is a circle with center at the origin and radius $c-1$, for $c \geq 1$. The values for $c < 1$ do not yield level curves because $\sqrt{x^2+y^2} = c-1$ is then impossible.

The contour diagram consists of concentric circles with appropriate labels. For instance you may pick the following three levels:

- *$c = 2$ which gives a circle of radius 1 with center at the origin. This circle is labeled by the level 2;*
- *$c = 3$ which gives a circle of radius 2 labeled by the level value 3;*
- *$c = 4$ which gives a circle of radius 3 labeled by the level value 4.*

The smallest value one can give c is $c = 1$ and that corresponds to a circle of radius 0 hence to a point $(0,0)$. The level diagram should show the level curves but also the levels. There are other choices for levels, so other answers may be correct as well.

(ii) *The graph is an upside-down cone with vertex at $(0,0,1)$.*

Problem 3.

Consider the points $P(3,0,-1)$, $Q(1,1,0)$ and $R(-1,1,2)$.

(i) Find the equation of the plane through P, Q and R.
(ii) Find the area of the triangle PQR.

`Answer:`

(i) *We have* $\vec{P}Q = (-2,1,1)$ *and* $\vec{P}R = (-4,1,3)$. *The cross product is normal to the plane* PQR. *We have*

$$\vec{P}Q \times \vec{P}R = (-2,1,1) \times (-4,1,3) = (2,2,2).$$

The plane is

$$2(x-3) + 2y + 2(z+1) = 0 \implies x + y + z = 2.$$

(ii) *The are of the triangle is given by half of the area of the parallelogram spanned by* $\vec{P}Q$ *and* $\vec{P}R$. *This is*

$$\frac{1}{2}||\vec{P}Q \times \vec{P}R|| = \frac{1}{2}||(2,2,2)|| = \frac{1}{2} \cdot \sqrt{2^2 + 2^2 + 2^2} = \frac{1}{2}\sqrt{12} = \sqrt{3}.$$

Problem 4.

Consider the plane P with equation $z = 6x - 3y + 2$.

(i) Find the equation of a plane parallel to P and passing through the point $(1, 0, -1)$.
(ii) For which value of a is the vector $(-2, 1, a)$ normal to the plane?

`Answer:`

(i) *We write the plane into standard form*

$$6x - 3y - z = -2.$$

The normal vector is $(6, -3, -1)$. *A plane parallel to* P *has the same normal vector* $(6, -3, -1)$. *Since it passes through* $(1, 0, -1)$ *we must have*

$$6(x - 1) - 3y - (z + 1) = 0 \implies 6x - 3y - z = 7.$$

(ii) *A normal vector is* $(6, -3, -1)$ *as determined in (i). For the vector* $(-2, 1, a)$ *to be normal, it has to be parallel to* $(6, -3, -1)$, *hence it has to be a multiple of this vector. This means the components are proportional. Comparing ratios of corresponding components we find*

$$\frac{6}{-2} = \frac{-3}{1} = \frac{-1}{a} \implies a = \frac{1}{3}.$$