Math 10C - Winter 2010 - Midterm II

Name: ____________________________

Student ID: _______________________

Section time: ______________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books or telephones. You may use a ”cheat sheet” of notes which should be a page, front only.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 4 questions which are worth 45 points. You have 50 minutes to complete the test.

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Problem 1. [11 points.]

Consider the function 
\[ f(x, y) = e^{-y}x^3. \]

(i) [3] Find the equation of the tangent plane to the graph \( z = f(x, y) \) at \( (1, 0, 1) \).

We compute 
\[
\begin{align*}
  f_x &= 3x^2e^{-y} \implies f_x(1, 0) = 3 \\
  f_y &= -x^3e^{-y} \implies f_y(1, 0) = -1.
\end{align*}
\]

The tangent plane is 
\[ z - 1 = 3(x - 1) - y \implies z = 3x - y - 2. \]

(ii) [3] Using linear approximation, estimate \( f(1.01, .01) \).

We use part (i) to estimate 
\[ z = f(1.01, .01) \approx 3 \cdot 1.01 - .01 - 2 = 1.02. \]
(iii) [5] Calculate the quadratic Taylor polynomial of \( f \) near \((1,0)\).

We calculate

\[
\begin{align*}
    f_x &= 3x^2e^{-y} \implies f_x(1,0) = 3 \\
    f_y &= -x^3e^{-y} \implies f_y(1,0) = -1 \\
    f_{xx} &= 6xe^{-y} \implies f_{xx}(1,0) = 6 \\
    f_{xy} &= -3x^2e^{-y} \implies f_{xy}(1,0) = -3 \\
    f_{yy} &= x^3e^y \implies f_{yy}(1,0) = 1.
\end{align*}
\]

The Taylor polynomial is

\[
1 + 3(x - 1) - y + 3(x - 1)^2 - 3(x - 1)y + \frac{1}{2}y^2.
\]
Problem 2. [12 points.]

Consider the function
\[ f(x, y) = x^2 \sin(2y - 2x) \]
and the point \( P(1, \frac{\pi}{2} + 1) \).

(i) [5] Find the gradient of \( f \) at the point \( P \).

We compute the derivatives
\[ f_x = 2x \sin(2y - 2x) - 2x^2 \cos(2y - 2x) \implies f_x(1, \frac{\pi}{2} + 1) = 2 \sin \pi - 2 \cos \pi = 2 \]
\[ f_y = 2x^2 \cos(2y - 2x) \implies f_y(1, \frac{\pi}{2} + 1) = 2 \cos \pi = -2. \]

Then
\[ \nabla f(P) = (2, -2). \]

(ii) [4] Calculate the directional derivative of \( f \) at \( P \) in the direction
\[ \vec{u} = \frac{3i + 4j}{5}. \]

We have
\[ f_{\vec{u}}(P) = \nabla f(P) \cdot \vec{u} = (2, -2) \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = 2 \cdot \frac{3}{5} - 2 \cdot \frac{4}{5} = -\frac{2}{5}. \]
(iii) [3] Find the (unit) direction of steepest increase for the function \( f(x, y) \) at \( P \).

We have
\[
\vec{u} = \frac{\nabla f}{||\nabla f||} = \frac{(2, -2)}{\sqrt{2^2 + (-2)^2}} = \left( \frac{2}{\sqrt{8}}, -\frac{2}{\sqrt{8}} \right) = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).
\]

Problem 3. [13 points.]

Consider the planes
\[ x + 5y + 3z = 1 \text{ and } 2x - y + z = 2. \]

(i) [3] For each of the planes, write down a normal vector.

\[
\vec{n}_1 = (1, 5, 3), \quad \vec{n}_2 = (2, -1, 1).
\]
(ii) [3] Are the two planes perpendicular?

We compute
\[ \vec{n}_1 \cdot \vec{n}_2 = (1, 5, 3) \cdot (2, -1, 1) = 1 \cdot 2 + 5 \cdot (-1) + 3 \cdot 2 = 0. \]
Therefore the two normal vectors, hence the two planes, are perpendicular.

(iii) [4] Find a vector parallel to the intersection of the two planes.

The vector parallel to the intersection of the two planes is normal to both \( \vec{n}_1 \) and \( \vec{n}_2 \). We can use the cross product to find such a vector
\[ \vec{n}_1 \times \vec{n}_2 = (1, 5, 3) \times (2, -1, 1) = (8, 5, -11). \]

(iv) [3] Write down a plane parallel to the plane \( x + 5y + 3z = 1 \) and passing through \((1, -1, 1)\).

The normal vector to the new plane is still \((1, 5, 3)\). The plane is
\[ (x - 1) + 5(y + 1) + 3(z - 1) = 0 \implies x + 5y + 3z = -1. \]
Problem 4. [9 points.]

Consider the function

\[ z = y^2 \ln x \]

and assume

\[ x = e^{2u}, \quad y = v^2 \sqrt{u}. \]

Using the chain rule, calculate the derivatives

\[ \frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v}. \]

Please express your answer in the simplest form.

We have

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.
\]

We compute

\[
\frac{\partial z}{\partial x} = y^2 \frac{\partial}{\partial x} \ln x = v^4u e^{2u},
\]

\[
\frac{\partial z}{\partial y} = 2y \ln x = 2v^2 \sqrt{u} \ln e^{2u} = 4v^2 u \sqrt{u},
\]

\[
\frac{\partial x}{\partial u} = 2e^{2u},
\]

\[
\frac{\partial y}{\partial u} = \frac{v^2}{2\sqrt{u}}.
\]

We get

\[
\frac{\partial z}{\partial u} = \frac{v^4 u}{e^{2u}} \cdot 2e^{2u} + 4v^2 u \sqrt{u} \cdot \frac{v^2}{2\sqrt{u}} = 2v^4 u + 2v^4 u = 4v^4 u.
\]

Similarly,

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = 4v^2 u \sqrt{u} \cdot 2v \sqrt{u} = 8v^3 u^2.
\]