Problem 1.

Consider differentiable functions \( f_n : [0, 1] \to \mathbb{R} \) for \( n \geq 1 \), such that the sequences \( \{f_n\} \) and \( \{f'_n\} \) are uniformly bounded. Show that \( \{f_n\} \) has a uniformly convergent subsequence.

Solution: By Arzelà-Ascoli, we only need to prove \( \{f_n\} \) is an equicontinuous family. This, together with boundedness, guarantees the existence of a uniformly convergent subsequence.

Let \( M > 0 \) such that

\[
|f'_n(x)| \leq M
\]

for all \( x \in [0, 1] \) and all \( n \geq 1 \). Fix \( \epsilon > 0 \). If \( x, y \in [0, 1] \) with

\[
|x - y| < \frac{\epsilon}{M}
\]

then for some \( \xi \in (x, y) \) we have

\[
|f_n(x) - f_n(y)| = |f'_n(\xi)(x - y)| \leq M|x - y| < \epsilon,
\]

for all \( n \). This confirms equicontinuity.
Problem 2.

Let $0 < a < \pi$. Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a. \end{cases}$$

(i) Find the Fourier coefficients $c_n$ of the function $f$. Be careful when calculating the Fourier coefficient $c_0$. Use Parseval’s theorem for the Fourier coefficients of $f$ to evaluate the sum

$$\sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2}.$$

(ii) Using the value $a = \frac{\pi}{2}$ in (i), calculate the sum $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Solution:

(i) We find the zeroth Fourier coefficient first

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-a}^{a} \, dx = \frac{a}{\pi}.$$

For $n \neq 0$, we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_{-a}^{a} e^{-inx} \, dx = \frac{1}{2\pi} \cdot \frac{1}{-in} e^{-inx} \bigg|_{x=a}^{x=-a} = \frac{1}{-2\pi in} (e^{-ina} - e^{ina}) = \frac{\sin(na)}{\pi n}.$$

We know

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_n |c_n|^2.$$

We compute

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-a}^{a} \, dx = \frac{a}{\pi}.$$

Also,

$$\sum_n |c_n|^2 = |c_0|^2 + 2 \sum_{n \geq 1} |c_n|^2 = \left(\frac{a}{\pi}\right)^2 + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2}.$$

From here, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin^2(na)}{n^2} = \frac{a(\pi - a)}{2}.$$

(ii) Making $a = \frac{\pi}{2}$ gives

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/2)}{n^2} = \pi^2 - \frac{8}{8}.$$

When $n$ is even, $\sin(n\pi/2) = 0$. When $n$ is odd, $\sin(n\pi/2) = \pm 1$. Thus, the above identity becomes

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \implies \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$
Problem 3.

Show that there does not exist a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)x^n \, dx = \frac{1}{n} \text{ for all } n \geq 1.$$

**Solution:** Consider the continuous function $g(x) = xf(x) - 1$. For $n \geq 0$, we calculate

$$\int_0^1 g(x)x^n \, dx = \int_0^1 (xf(x) - 1)x^n \, dx = \int_0^1 f(x)x^{n+1} \, dx - \int_0^1 x^n \, dx = \frac{1}{n+1} - \int_0^1 x^n \, dx = 0.$$

We have seen in the homework that this implies that $g(x) = 0$. (The proof used the Weierstrass theorem.) However,

$$g(0) = 0 \cdot f(0) - 1 = -1$$

which is a contradiction.
Problem 4.

Consider the series

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^3 x}. \]

(i) Show that the series converges uniformly for all \( x \in [a, \infty), \) for all \( a > 0. \) Conclude that the series converges pointwise for all \( x \in (0, \infty). \)

(ii) Show that the convergence is not uniform over the interval \((0, \infty).\)

(iii) Show that the sum of the series is differentiable over any interval \([a, \infty)) \) with \( a > 0. \) Conclude that \( f \) is differentiable also over \((0, \infty).\)

Solution:

(i) Fix \( a > 0. \) We apply the Weierstraß M-test to the functions

\[ f_n(x) = \frac{1}{1 + n^3 x}. \]

over the interval \([a, \infty).\) For each \( x \geq a, \) we have

\[ |f_n(x)| \leq \frac{1}{n^3 x} < \frac{1}{n^3 a}. \]

For \( M_n = \frac{1}{n^3 a} \) we have

\[ \sum_{n=1}^{\infty} M_n = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 < \infty. \]

By the Weierstraß M-test, the series converges uniformly over \([a, \infty).\)

The convergence is also pointwise over the interval \([a, \infty).\) We claim pointwise convergence over \((0, \infty). \) Fix \( x > 0. \) We can find \( a > 0 \) such that \( x \in [a, \infty), \) so it follows that the series converges pointwise at any \( x > 0. \)

(ii) We check that Cauchy’s criterion is not satisfied by considering the difference between two consecutive partial sums. Consider the partial sums

\[ s_n = f_1 + \ldots + f_n. \]

If the convergence is uniform over \((0, \infty),\) for any \( \epsilon > 0 \) we can find \( N \) such that

\[ ||s_n - s_m|| \leq \epsilon \]

for \( n, m \geq N, \) where \( || \cdot || \) means supremum norm. Take \( \epsilon = \frac{1}{3}. \) Put \( m = n - 1. \) Then, for \( n > N \) we have

\[ ||s_n - s_{n-1}|| \leq \frac{1}{3} \Rightarrow ||f_n|| \leq \frac{1}{3}. \]

This is however false since

\[ f_n \left( \frac{1}{n^3} \right) = \frac{1}{2} > \frac{1}{3}. \]

Thus the convergence is not uniform over \((0, \infty).\)
We show that \( f \) is differentiable over the interval \([a, \infty)\) for \( a > 0 \), and that
\[
f'(x) = -\sum_{n=1}^{\infty} \frac{n^3}{(1 + n^3x)^2}
\]
over \([a, \infty)\). Since the intervals \([a, \infty)\) with \( a > 0 \) cover \((0, \infty)\), it follows that \( f \) is differentiable over \((0, \infty)\).

To this end, consider the partial sums
\[
s_n = f_1 + \ldots + f_n
\]
and note that \( s_n \) are differentiable, and furthermore
\[
s'_n = f'_1 + \ldots + f'_n.
\]
We claim that the derivatives \( s'_n \) converge uniformly to some function \( g \) over the interval \([a, \infty)\) i.e.
\[
g(x) = -\sum_{n=1}^{\infty} \frac{n^3}{(1 + n^3x)^2}.
\]
Since \( s_n \) converges to \( f \), it follows that \( f \) is differentiable and \( f' = g \).

To prove the claim, note that
\[
f'_n = -\frac{n^3}{(1 + n^3x)^2}.
\]
We apply again the Weierstraß \( M \)-test, noting
\[
|f'_n| \leq \frac{n^3}{(n^3x)^2} \leq \frac{1}{n^3a^2}.
\]
Since
\[
\sum_{n=1}^{\infty} \frac{1}{n^3a^2}
\]
converges for \( a > 0 \), the partial sums \( s'_n \) converge as well, establishing the claim.
Problem 5.

Consider a differentiable function $f : [0, 1] \to \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in [0, 1]$. Show that for all $n \geq 1$, we have

$$\left| \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) - \int_{0}^{1} f(x) \, dx \right| \leq \frac{M}{n}.$$

Use this result to calculate the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin \left( \frac{k\pi}{n} \right).$$

Solution: We break the integral into $n$ sub-integrals over the intervals $[\frac{k-1}{n}, \frac{k}{n}]$ for $1 \leq k \leq n$. We have

$$\left| \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) - \int_{0}^{1} f(x) \, dx \right| = \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f \left( \frac{k}{n} \right) \, dx - \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x) \, dx \right|$$

$$= \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( f \left( \frac{k}{n} \right) - f(x) \right) \, dx \right|$$

$$\leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| f \left( \frac{k}{n} \right) - f(x) \right| \, dx.$$

By the mean value theorem, for all $x \in [\frac{k-1}{n}, \frac{k}{n}]$ we can find $\xi \in [\frac{k-1}{n}, \frac{k}{n}]$ such that

$$\left| f \left( \frac{k}{n} \right) - f(x) \right| = \left| \frac{k}{n} - x \right| \cdot |f'(\xi)| \leq \frac{1}{n} \cdot M.$$

Substituting in the above, we find

$$\left| \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) - \int_{0}^{1} f(x) \, dx \right| \leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{M}{n} \, dx = \int_{0}^{1} M \frac{1}{n} \, dx = \frac{M}{n}.$$

Thus,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) = \int_{0}^{1} f(x) \, dx.$$

We apply this result to the function

$$f(x) = \sin(\pi x).$$

We have $|f'(x)| = |\pi \cos(\pi x)| \leq \pi$, so the hypothesis is satisfied. Also,

$$\int_{0}^{1} \sin(\pi x) \, dx = \frac{2}{\pi}.$$

Thus,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin \left( \frac{k\pi}{n} \right) = \frac{2}{\pi}.$$
Problem 6.

Assume that $f_n : [0, 1] \to \mathbb{R}$ are continuous functions converging uniformly to $f : [0, 1] \to \mathbb{R}$. Show that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.$$

Solution: Since the convergence is uniform, we already proved in class that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.$$

We show that

$$\lim_{n \to \infty} \int_{1 - \frac{1}{n}}^1 f_n(x) \, dx = 0.$$

Subtracting, we find

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx - \lim_{n \to \infty} \int_{1 - \frac{1}{n}}^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.$$

This rewrites as

$$\lim_{n \to \infty} \int_0^{1 - \frac{1}{n}} f_n(x) \, dx = \int_0^1 f(x) \, dx,$$

as claimed.

To prove

$$\lim_{n \to \infty} \int_{1 - \frac{1}{n}}^1 f_n(x) \, dx = 0,$$

we estimate the integrals above using the supremum norm $\| \cdot \|$. We have

$$\left| \int_{1 - \frac{1}{n}}^1 f_n(x) \, dx \right| \leq \int_{1 - \frac{1}{n}}^1 |f_n(x)| \, dx \leq \int_{1 - \frac{1}{n}}^1 \|f_n\| \, dx = \frac{1}{n} \|f_n\|.$$

We claim $\frac{1}{n} \|f_n\| \to 0$ as $n \to \infty$. Thus, by the squeeze theorem

$$\lim_{n \to \infty} \int_{1 - \frac{1}{n}}^1 f_n(x) \, dx = 0,$$

as needed.

To prove the last claim, note that by the triangle inequality

$$\|f_n\| \leq \|f_n - f\| + \|f\| \implies \frac{1}{n} \|f_n\| \leq \frac{1}{n} \|f_n - f\| + \frac{1}{n} \|f\| \to 0,$$

as $n \to \infty$ since $\|f_n - f\| \to 0$. 
Problem 7.

Consider the function \( f : [0, 1] \rightarrow \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \\
0 & \text{otherwise}
\end{cases}
\]

(i) What is the value of the lower Riemann sum \( L(P, f) \) for any partition \( P \)?

(ii) Show that \( f \) is Riemann integrable.

Solution:

(i) For any partition \( P \), we have \( L(P, f) = 0 \) since the infimum of \( f \) over any interval is 0.

(ii) We show that for any \( \epsilon > 0 \) we can find a partition \( P \) with

\[
U(P, f) < \epsilon.
\]

This will prove \( f \) is integrable and

\[
\int_0^1 f(x) \, dx = 0.
\]

To this end, pick a positive integer \( k \) sufficiently large so that

\[
\frac{1}{k} < \frac{\epsilon}{2}.
\]

We construct the partition \( P \) as follows:

(a) the first interval of the partition is \( [0, \frac{1}{k}] \);

(b) around the points \( 1, \frac{1}{2}, \ldots, \frac{k-1}{k} \) construct small intervals of length \( \frac{\epsilon}{2k} \) containing these points. Shrink the intervals if necessary so that they don’t overlap;

(c) fill in the partition with the gap intervals so that together with (a) and (b) we cover \([0, 1]\).

We estimate \( U(P, f) \) for the partition thus constructed:

(a) for the first interval (a),

\[
\sup f \leq 1
\]

and the interval length is \( \frac{1}{k} \). Its contribution to the sum \( U(P, f) \) is at most

\[
\frac{1}{k} < \frac{\epsilon}{2};
\]

(b) for the \( k \) intervals in (b), the supremum of \( f \) over these intervals is 1, while each length interval is at most \( \frac{\epsilon}{2k} \). These \( k \) intervals contribute to \( U(P, f) \) at most

\[
k \cdot \frac{\epsilon}{2k} = \frac{\epsilon}{2};
\]

(c) the intervals (c) have no contribution to \( U(P, f) \) since the supremum of \( f \) is zero.

Adding these contributions together, we find

\[
U(P, f) < \frac{\epsilon}{2} + \frac{\epsilon}{2} + 0 = \epsilon.
\]
Problem 8.

Assume that

\[ f : \mathbb{R} \to \mathbb{C} \]

is a \(2\pi\)-periodic function of class \(C^2\) i.e. the second derivative \(f''\) exists and is continuous. Show that the Fourier series of \(f\) converges uniformly to the function \(f\), following the steps below.

Let \(c_n\) denote the Fourier coefficients of \(f\).

(i) Using integration by parts, show that the Fourier coefficients of \(f''\) equal \(-n^2c_n\).

(ii) Using (i), explain that \(n^2c_n \to 0\). Prove that there exists a constant \(M > 0\) such that \(|c_n| \leq \frac{M}{n^2}\) for all \(n \neq 0\).

(iii) Using the Weierstraß \(M\)-test, show that the Fourier series of \(f\) converges uniformly. That is, show that the partial sums

\[ s_N = \sum_{n=-N}^{N} c_n e^{inx} \]

converge uniformly to some continuous function \(g\), as \(N \to \infty\).

(iv) It remains to show that \(f = g\). Using part (iii) and integration, show that the \(k\)th Fourier coefficients of \(f\) and \(g\) agree, for all \(k\).

(v) Show that if the Fourier coefficients of two continuous functions \(f\) and \(g\) agree, then \(f = g\).

Solution:

(i) The Fourier coefficients of \(f''\) equal

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f''(x)e^{-inx} \, dx = \frac{1}{2\pi} \left( f'(x)e^{-inx}\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)(-in)e^{-inx} \, dx \right)
= \frac{-in}{2\pi} \int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx
= \frac{-in}{2\pi} \left( f(x)e^{-inx}\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-in)e^{-inx} \, dx \right)
= \frac{(in)^2}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx
= -n^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx = -n^2c_n.
\]

In the above calculations we used the periodicity of \(f\) and \(f'\) to cancel the boundary-terms in the integration by parts formula.

(ii) Since \(f''\) is continuous, it is integrable. The Fourier coefficients of any integrable function converge to 0, as shown in class. Thus

\[ n^2c_n \to 0 \]
as \( n \to \infty \). In particular, \( n^2c_n \) is a bounded sequence, hence

\[
|c_n| \leq \frac{M}{n^2}
\]

for some constant \( M > 0 \), and all \( n \neq 0 \).

(iii) We apply the Weierstraß M-test to the functions \( c_ne^{inx} \). We have

\[
|c_ne^{inx}| = |c_n| \leq \frac{M}{n^2}
\]

and

\[
\sum_{n\neq 0} \frac{M}{n^2} < \infty.
\]

Thus,

\[
s_N = \sum_{n=-N}^{N} c_ne^{inx}
\]

converges uniformly (the \( n = 0 \) term is constant and does not affect convergence). The limit function \( g \) is continuous since each \( s_N \) is continuous and the convergence is uniform.

(iv) Fix an integer \( k \). We show that the \( k \)th Fourier coefficients of \( f \) and \( g \) agree. We claim first that

\[
s_N(x)e^{-ikx} \to g(x)e^{-ikx}
\]

uniformly as \( N \to \infty \) over \([-\pi, \pi]\). Indeed, in supremum norm

\[
||s_Ne^{-ikx} - g(x)e^{-ikx}|| = ||s_N - g|| \to 0
\]

as \( N \to \infty \), by part (iii). Here, the term \( e^{-ikx} \) does not affect the supremum norm since its absolute value is 1.

Integrating, we find that as \( N \to \infty \),

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(x)e^{-ikx} \, dx \to \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)e^{-ikx} \, dx.
\]

We calculate the left hand side. For \( N \geq k \), we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(x)e^{-ikx} \, dx = \sum_{n=-N}^{N} c_n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-k)x} \, dx = c_k
\]

where we have used the orthogonality noted in class

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} \, dx = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}
\]

Substituting in the limit, we obtain that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)e^{-ikx} \, dx = c_k
\]

i.e. the \( k \)th Fourier coefficients of \( g \) and \( f \) agree.
(v) The Fourier coefficients of $h = f - g$ are equal to 0 by (iv). By Parseval’s theorem

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 \, dx = 0.$$  

Since $h$ is continuous ($f$ and $g$ both are), it follows $h = 0$ over $[-\pi, \pi]$ using Rudin, chapter 6, exercise 2. By periodicity, $h = 0$ everywhere, hence $f = g$. 