Name: ________________________________

Student ID: ____________________________

Instructions:

There are 4 questions which are worth 40 points. You have 50 minutes to complete the test.

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Problem 1. [10 points.]

Consider a twice differentiable function $f : [-1, 1] \to \mathbb{R}$ such that

$$f(-1) = -1, f(0) = 0, f(1) = 3.$$ 

(i) [3] Give an example of a degree 2 polynomial $P(x) = x^2 + ax + b$ which satisfies these three properties.

(ii) [7] Comparing $f$ to the polynomial $P$, show that there exists a value $x_0 \in (-1, 1)$ such that $f''(x_0) = 2$. 
Problem 2. \([10 \text{ points}.]\]

Find the values of \(\alpha\) such that the function \(f : \mathbb{R} \to \mathbb{R}\) given by

\[
f(x) = \begin{cases} 
  x^{\alpha} \ln x & \text{for } x > 0, \\
  0 & \text{for } x \leq 0
\end{cases}
\]

is differentiable.
Problem 3. [10 points.]

(i) [4] Briefly state what it means for a bounded function \( f : [a, b] \to \mathbb{R} \) to be Riemann integrable (with respect to \( \alpha(x) = x \)).

(ii) [6] Prove that if \( f : [a, b] \to \mathbb{R} \) is an increasing function, then \( f \) is Riemann integrable (with respect to \( \alpha(x) = x \)).

You may use any of the results we proved in class, but merely stating “it’s a theorem in Rudin” will not be enough.
Problem 4. [10 points.]

Let $f : [0, 2] \rightarrow \mathbb{R}$ be a differentiable function with

$$f(0) = 0, f(2) = 0, |f'(x)| \leq 1 \text{ for all } x \in [0, 1].$$

Follow the steps below to prove that

$$\left| \int_0^2 f(t) \, dt \right| \leq 1.$$

(i) [2] Show that for all $t \in [0, 1]$ we have $|f(t)| \leq t$. (Hint: Mean value theorem)

(ii) [2] Similarly, show that for all $t \in [1, 2]$ we have $|f(t)| \leq 2 - t$. 


(iii) [4] Using (i) and (ii), show that
\[
\left| \int_{0}^{2} f(t) \, dt \right| \leq 1.
\]

(iv) [2] Can equality occur in (iii)?