Review Problems for Midterm II

Make sure you review all homework problems, definitions, theorems and proofs.

Problem 1.

Consider $C^1([a, b])$ the set of continuously differentiable real valued functions endowed with the norm

$$||f|| = \sup|f(x)| + \sup |f'(x)|.$$ 

(i) Show that $C^1[a, b]$ is a metric space via the metric $d(f, g) = ||f - g||$.

(ii) Is $C^1([a, b])$ complete?

Problem 2.

Show that if $\mathcal{F}$ is a family of differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with uniformly bounded derivatives $|f'| \leq M$ for all $f \in \mathcal{F}$ then $\mathcal{F}$ is equicontinuous. In particular, show that the sequence $f_n(x) = \frac{\sin(nx)}{n}$ forms an equicontinuous family.

Problem 3.

Show that an equicontinuous family of pointwise bounded sequence of functions defined on a compact set is uniformly bounded.

Problem 4.

Solve Problem 20, Chapter 7 of Rudin.

Problem 5.

Solve Problem 5, Chapter 7 of Rudin.

Problem 6.

Assume that $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x) \, dx = \int_0^1 f(x) \left(x^n + x^{n+2}\right) \, dx$$

for all $n$.

(i) Make $n \rightarrow \infty$ to conclude that $\int_0^1 f(x) \, dx = 0$. 

(ii) Use Problem 4, to conclude that in fact \( f = 0 \).

**Problem 7.**

Prove that the series
\[
\sum_{n=1}^{\infty} \frac{n x^2}{n^3 + x^3}
\]
converges uniformly on \([0, a]\) for all \( a > 0 \), and that
\[
\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{n x^2}{n^3 + x^3} = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}.
\]

**Problem 8.**

Let \( f: [0, \frac{1}{2}] \to \mathbb{R} \) be continuous. Show that
\[
\lim_{n \to \infty} \int_{0}^{\frac{1}{2}} f(x^n) \, dx = \frac{f(0)}{2}.
\]

**Problem 9.**

Consider
\[
f_n(x) = x(1 - x)^n.
\]

(i) Show that \( f_n \) converges uniformly on \([0, 1]\) and find the limit.
(ii) Show that \( \sum_n f_n \) converges over \([0, 1]\) pointwise, but not uniformly.

**Problem 10.**

Assume that \( f: [0, 1] \to \mathbb{R} \) is a continuous function. Show that the sequence \( x^n f(x) \) converges uniformly as \( n \to \infty \) if and only if \( f(1) = 0 \).

**Problem 11.**

Prove that the series
\[
\sum_{n=0}^{\infty} 2^n \sin \frac{1}{3^n x}
\]
converges uniformly over \([1, \infty)\).

**Problem 12.**

Prove that the series
\[
\sum_{n=0}^{\infty} \frac{n x}{1 + n^4 x^2}
\]
converges uniformly over \([1, \infty)\).