Math 140B - Spring 2017 - Final Exam

Name: ______________________________________

Student ID: ________________________________

Instructions:

Please print your name, student ID.

During the test, you may not use books, notes or telephones.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 7 questions which are worth 70 points total and the extra credit problem. You have 180 minutes to complete the test.

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Problem 1. [10 points; 3, 4, 3.]

Consider the sequence $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \frac{nx^2}{1 + nx^2}.$$

(i) Show that $f_n$ converges pointwise over $[0, \infty)$ and find the limit function $f$.

(ii) Show that $f_n$ converges uniformly over $[a, \infty)$ for all $a > 0$.

(iii) Show that $f_n$ does not converge uniformly over $[0, \infty)$. 
Problem 2. [10 points.]

Assume that \( f, g : [a, b] \to \mathbb{R} \) are integrable functions. Show that \( f + g : [a, b] \to \mathbb{R} \) is also integrable.
**Problem 3.** [10 points; 6, 4.]

Consider the function

\[ f(x) = \begin{cases} 
-1 & \text{if } 0 \leq x < \pi \\
1 & \text{if } \pi \leq x < 2\pi 
\end{cases} . \]

(i) Compute the Fourier coefficients of \( f \).

(ii) Use (i) to determine the value of the sum

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} . \]
Problem 4. [10 points.]

Let $\mathcal{A}$ denote the set of all polynomials $P$ such that

$$P'(0) = 0.$$ 

Show that $\mathcal{A}$ is an algebra. Furthermore, exhibit some elements of $\mathcal{A}$, and use them to show that $\mathcal{A}$ separates points and vanishes nowhere.
Problem 5. [10 points.]

Let \( f_n : \mathbb{R} \to \mathbb{R} \) be a sequence of differentiable functions such that \( |f'_n(x)| \leq 1 \) for all \( x \in \mathbb{R} \). Assume that \( f_n \to g \) pointwise, for some function \( g : \mathbb{R} \to \mathbb{R} \). Show that \( g \) is continuous.
Problem 6. [10 points.]

Assume that $f_n : [a, b] \to \mathbb{R}$ is an equicontinuous sequence of functions. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a bounded, uniformly continuous function. Show that the sequence of functions

$$g_n : [a, b] \to \mathbb{R}, \quad g_n = \Phi \circ f_n$$

has a uniformly convergent subsequence.
Problem 7. [10 points; 6, 4.]

Consider the series
\[ f(x) = \sum_{n=0}^{\infty} \frac{nx}{1 + e^n x^2}. \]

(i) Show that \( f \) is differentiable over \((-\infty, -a) \cup (a, \infty)\) for all \( a > 0 \). Conclude that \( f \) is differentiable over \( \mathbb{R} \setminus \{0\} \).
(ii) Show that $f$ is not differentiable at $x = 0$. You may wish to start with the definition of differentiability.
Extra credit. [10 points.]

Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Assume that $\mathcal{A}$ is an algebra of continuous functions $\Phi : \mathbb{R} \to \mathbb{R}$ that separates points and vanishes nowhere. Show that

$$f \text{ is integrable } \iff \Phi \circ f \text{ is integrable for all } \Phi \in \mathcal{A}.$$