Problem 1.

Prove L'Hospital rule in the following form. Consider two differentiable functions

\[ f, g : (0, \infty) \to \mathbb{R} \]

such that \( g'(x) \neq 0 \) for all \( x \) and

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0.
\]

(i) Assuming that \( \lim_{x \to 0} \frac{f'(x)}{g'(x)} = L \), show that \( \lim_{x \to 0} \frac{f(x)}{g(x)} = L \).

(ii) By means of a counterexample, show that the converse of (i) is false.

Explicitly, consider the functions

\[ f(x) = x^2 \sin \left( \frac{1}{x} \right), \quad g(x) = x. \]

Show that \( \lim_{x \to 0} \frac{f(x)}{g(x)} = 0 \) but \( \lim_{x \to 0} \frac{f'(x)}{g'(x)} \neq 0 \). (In fact, the latter limit does not exist.)

Solution:

(i) The proof of this statement was given in class. Even though \( f \) and \( g \) are initially undefined at 0, we may set

\[ f(0) = g(0) = 0. \]

With this choice of values for \( f(0) \) and \( g(0) \), we note that the hypotheses rewrite

\[
\lim_{x \to 0} f(x) = f(0), \quad \lim_{x \to 0} g(x) = g(0).
\]

In particular \( f, g \) are continuous at 0. Using differentiability over \((0, \infty)\), it follows that \( f, g \) are continuous over any interval \([0, x]\) and differentiable over \((0, x)\), for any value of \( x > 0 \). This means that \( f, g \) are Rolle functions over \([0, x]\), and we may apply Cauchy’s mean value theorem. Thus, there exists \( c \in (0, x) \) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f(x)}{g(x)}.
\]

Note that by assumption \( g'(c) \neq 0 \) so the first fraction is well-defined. Similarly, the second fraction makes sense since \( g(x) \neq 0 \). (Otherwise, if \( g(x) = 0 \) then Rolle’s theorem for \( g \) over the interval \([0, x]\) would produce an intermediate point \( d \) such that \( g'(d) = 0 \) which is not allowed.)

We make \( x \to 0 \) in the identity above. In particular, we also have \( c \to 0 \) since \( c \in (0, x) \). By hypothesis the left hand side converges to \( L \). Thus, the same is true for the right hand side

\[
L = \lim_{c \to 0} \frac{f'(c)}{g'(c)} = \lim_{x \to 0} \frac{f(x)}{g(x)}.
\]
(ii) By direct computation,
\[ \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0. \]

Indeed, this limit follows by the squeeze theorem since
\[ \left| x \sin \left( \frac{1}{x} \right) \right| \leq |x| \to 0 \text{ as } x \to 0. \]

Clearly, \( g'(x) \neq 0 \), and \( \lim_{x \to 0} g(x) = 0 \). Furthermore,
\[ \left| x^2 \sin \left( \frac{1}{x} \right) \right| \leq |x^2| \to 0 \text{ as } x \to 0 \implies \lim_{x \to 0} f(x) = 0. \]

Finally, we compute
\[ \frac{f'(x)}{g'(x)} = 2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right). \]

To see this does not converge to 0, pick the sequence \( x_n = \frac{1}{2\pi n} \to 0 \). Note that by choice, \( \sin \left( \frac{1}{x_n} \right) = 0 \) and \( \cos \left( \frac{1}{x_n} \right) = 1 \). The values
\[ \frac{f'(x_n)}{g'(x_n)} = -1 \]
show convergence to 0 is impossible.
Problem 2.

Prove the following “higher order” derivative test from calculus.

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function and let \( k \geq 1 \). Assume that \( f \) admits the first \( 2k \) derivatives, with \( f^{(2k)} \) continuous. Assume that

\[
f'(0) = \ldots = f^{(2k-1)}(0) = 0.
\]

(In particular, since \( f'(0) = 0 \), it follows that 0 is a critical point for \( f \).) Moreover, assume that

\[
f^{(2k)}(0) > 0.
\]

Prove that \( f \) has a local minimum at 0. The case \( k = 1 \) corresponds to the “second derivative test.”

(i) Write down Taylor’s theorem for the function \( f \) around 0 up to order \( 2k \).
(ii) Using (i), show that 0 is a local minimum.

Solution:

(i) By Taylor’s theorem for \( f \), for any \( x \) there exists an intermediate point \( c \) between 0 and \( x \) such that

\[
f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(2k-1)}(0)}{(2k-1)!}x^{2k-1} + \frac{f^{(2k)}(c)}{(2k)!}x^{2k}.
\]

Using that

\[
f'(0) = \ldots = f^{(2k-1)}(0) = 0
\]

we conclude that

\[
f(x) = f(0) + \frac{f^{(2k)}(c)}{(2k)!}x^{2k}.
\]

(ii) We wish to find an interval \((-\delta, \delta)\) such that for all \( x \in (-\delta, \delta) \) we have

\[
f(x) \geq f(0) \text{ which by (i) is equivalent to } f^{(2k)}(c) \geq 0.
\]

Furthermore, if \( x \in (-\delta, \delta) \), it also follows that \( c \in (-\delta, \delta) \). It suffices to explain that we can find \( \delta \) so that for all \( c \in (-\delta, \delta) \) we have \( f^{(2k)}(c) \geq 0 \).

To specify \( \delta \), we use that \( f^{(2k)}(0) > 0 \) and \( f^{(2k)} \) is continuous so that \( f^{(2k)} \) is positive in a neighborhood of 0. More rigorously, set

\[
\epsilon = \frac{1}{2} f^{(2k)}(0) > 0.
\]

There exists \( \delta > 0 \) such that if \( c \in (-\delta, \delta) \) we have

\[
|f^{(2k)}(c) - f^{(2k)}(0)| < \epsilon = \frac{1}{2} f^{(2k)}(0) \quad \Rightarrow \quad -\epsilon < f^{(2k)}(c) - f^{(2k)}(0) \quad \Rightarrow \quad \frac{1}{2} f^{(2k)}(0) < f^{(2k)}(c) - f^{(2k)}(0) \quad \Rightarrow \quad \frac{1}{2} f^{(2k)}(0) < f^{(2k)}(c).
\]

Since \( f^{(2k)}(0) > 0 \) we obtain \( f^{(2k)}(c) > 0 \) for all \( c \in (-\delta, \delta) \).
Problem 3.

Let \( f: [a, b] \rightarrow \mathbb{R} \), and let \( c \in (a, b) \). Assume that \( f \) is integrable over the intervals \([a, c]\) and \([c, b]\) respectively.

(i) Show that \( f \) is integrable over \([a, b]\).

(ii) Show that
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

Solution:

(i) Fix \( \epsilon > 0 \). Since \( f \) is integrable over \([a, c]\) there exists a partition \( P \) of \([a, c]\) such that
\[
U(P, f) - L(P, f) < \frac{\epsilon}{2}.
\]
Similarly, since \( f \) is integrable over \([c, 1]\) we can find a partition \( Q \) of \([c, b]\) such that
\[
U(Q, f) - L(Q, f) < \frac{\epsilon}{2}.
\]
Let \( R \) be the partition obtained by putting together \( P \) and \( Q \). Thus \( R \) is a partition of \([a, b]\) and furthermore,
\[
U(R, f) - L(R, f) = (U(P, f) - L(P, f)) + (U(Q, f) - L(Q, f)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
This shows that \( f \) satisfies Riemann's criterion over \([a, b]\), hence \( f \) is integrable over \([a, b]\).

(ii) For simplicity, let us write \( K = \int_a^b f(x) \, dx \), \( I = \int_a^c f(x) \, dx \) and \( J = \int_c^b f(x) \, dx \). We will show that for all \( \epsilon > 0 \), we have
\[
-\epsilon < K - I - J < \epsilon
\]
which will then imply that \( K = I + J \) by making \( \epsilon \rightarrow 0 \).

Fix \( \epsilon > 0 \). By definition, \( I = \inf_P U(P, f) \) hence we can find a partition \( P \) of \([a, c]\) such that
\[
U(P, f) - I < \frac{\epsilon}{2}.
\]
Similarly, we can find \( Q \) a partition of \([c, b]\) such that
\[
U(Q, f) - J < \frac{\epsilon}{2}.
\]
Letting as above \( R = P \cup Q \) we obtain
\[
U(R, f) - I - J = (U(P, f) - I) + (U(Q, f) - J) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Note that \( K \leq U(R, f) \) for all partitions \( R \) since \( K = \inf_R U(R, f) \). Thus the above inequality gives
\[
K - I - J \leq U(R, f) - I - J < \epsilon
\]
which gives the claimed upper bound. The lower bound is obtained by working with the lower sums instead.