Math 140B - Lecture 25

May 27, 2022

We studied series & power series.

Today, we consider Fourier series.

Fourier series were introduced by J. Fourier to study the heat equation in the early 1800's.

Nowadays, they have applications to differential equations, image & audio processing, mathematical physics etc.

We only scratch the surface of Fourier analysis.

§1 Trigonometric polynomials

Definition A trigonometric polynomial is an expression

$$T(x) = \sum_{n=-N}^{N} c_n \, e^{inx}, \quad c_n \in \mathbb{C}$$

Remark Using $e^{inx} = \cos(nx) + i \sin(nx)$ any trigonometric polynomial can be expressed in terms of $\cos(nx)$, $\sin(nx)$

Remark

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \, dx = \begin{cases} 1 & , \; n = 0 \\ 0 & , \; n \neq 0 \end{cases}$$

Indeed, for $n \neq 0$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \, dx = \frac{1}{2\pi} \left. \frac{e^{inx}}{in} \right|_{x=-\pi}^{x=\pi} = 0 \quad \text{since} \quad e^{in\pi} = e^{-in\pi} = (-1)^n .$$

Remark We have

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(x)\, e^{-ikx}\, dx. \qquad \forall\ |k| \le N.$$

Indeed,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(x)\, e^{-ikx}\, dx = \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n\, e^{inx} \cdot e^{-ikx}\, dx$$

$$= \sum_{n=-N}^{N} c_n \cdot \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)x}\, dx}_{0 \text{ if } n \neq k \text{ or } 1 \text{ if } n = k}$$

$$= c_k$$ as claimed.

↳ previous remark.

<u>Definition</u> Trigonometric series are expressions of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

The N^{th} partial sum is defined to be

$$S_N = \sum_{n=-N}^{N} c_n e^{inx}$$

§2. Fourier series

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ integrable over $[-\pi, \pi]$ & 2π-periodic.

Definition The Fourier coefficients of f are defined by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx .$$

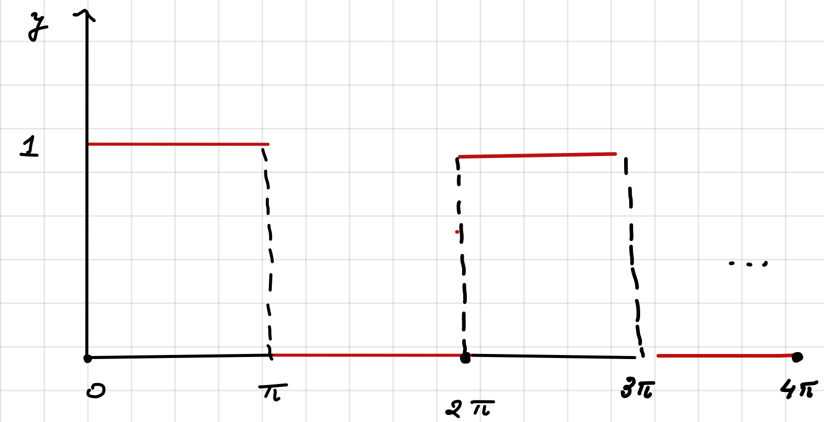
Notation

- $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$ = Fourier series of f.

We also write $f \sim \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$. This does not imply convergence of any sort (yet), it simply says RHS is the Fourier series of f.

Example Let

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ 0, & \pi \leq x < 2\pi \end{cases}$$



Extend $f : \mathbb{R} \longrightarrow \mathbb{C}$ by requiring f be 2π-periodic.

Fourier coefficients of f

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2\pi} \cdot x \Big|_{x=0}^{x=\pi} = \frac{1}{2}. \quad , \quad \boxed{c_0 = \frac{1}{2}}$$

For $n \neq 0$, we compute

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\cdot \frac{e^{-inx}}{-in}\Bigg|_{x=0}^{x=\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{e^{-in\pi}-1}{-in} = \frac{1}{2\pi}\cdot\frac{(-1)^n-1}{-in}, \quad \text{using } \boxed{e^{-in\pi} = e^{in\pi} = (-1)^n.}$$

Thus $\boxed{c_n = 0 \text{ for } n \text{ even} \neq 0},$

$$\boxed{c_n = \frac{1}{\pi i n} \text{ for } n \text{ odd.}}$$

$$\boxed{c_0 = \frac{1}{2}}$$

Question Does the Fourier series converge to f?

Answer is "no" for pointwise & uniform convergence, unless further assumptions are made about f. (Rudin 8.14).

We will not consider these.

Instead, we establish a general result. This requires a new notion of convergence.

§ 3. L^2- convergence

Recall that in Homework 3 we defined the "L^2-distance" to be

$$d_{L^2}(f,g) = \|f-g\|_{L^2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2\right)^{1/2}.$$

Definition $\mathcal{L}^2$- convergence is convergence in the metric d:

$$f_n \longrightarrow f \text{ in } \mathcal{L}^2 \iff d(f_n, f) \longrightarrow 0 \iff \lim_{n\to\infty} \int_{-\pi}^{\pi} |f_n(x) - f(x)|^2\, dx = 0.$$

Remark Over a general interval $[a,b]$, the distance is given by

$$d_{L^2}(f,g) = \|f-g\|_{L^2} = \left(\frac{1}{b-a}\int_a^b |f(x)-g(x)|^2\, dx\right)^{1/2}.$$

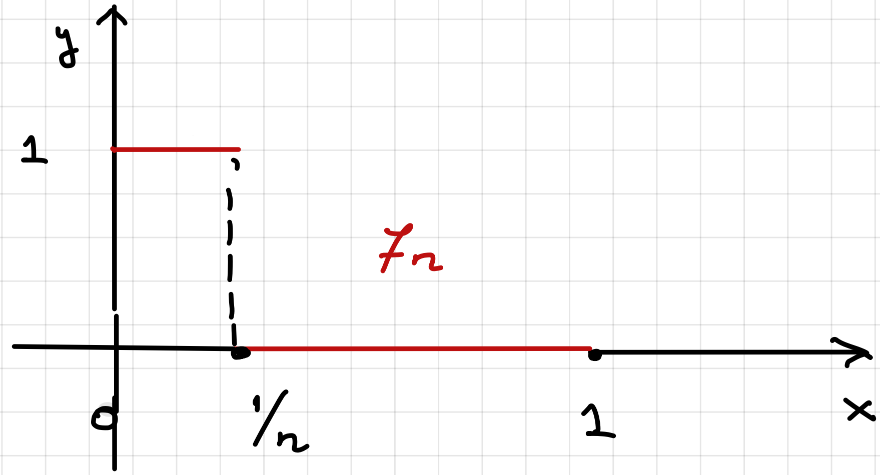
Remark $\boxed{i}$ If $f_n \rightrightarrows f$ in $[-\pi, \pi]$ then $f_n \xrightarrow{L^2} f$ in $[-\pi, \pi]$ (HWK 7, #5)

$\boxed{ii}$ Pointwise & L^2 convergence are harder to compare.

Example

$f_n : [0,1] \longrightarrow \mathbb{R}$

$$f_n(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{n} \\ 0, & \frac{1}{n} < x \le 1. \end{cases}$$



We let $f(x) = 0$. We show $f_n \longrightarrow f$ in $\mathcal{L}^2$.

Indeed,

$$\int_0^1 |f_n(x) - f(x)|^2 \, dx = \int_0^1 f_n(x)^2 \, dx = \int_0^{1/n} dx = \frac{1}{n} \longrightarrow 0 \quad \text{so}$$

$f_n \longrightarrow f$ in $\mathcal{L}^2$.

§4. Parseval's Theorem (8.16)

An important result in Fourier analysis is:

Theorem Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic, integrable over $[-\pi, \pi]$.

$\boxed{\text{i}}$ The Fourier series converges to f in L^2.

This means that if S_N are the Fourier partial sums then

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 \, dx = 0.$$

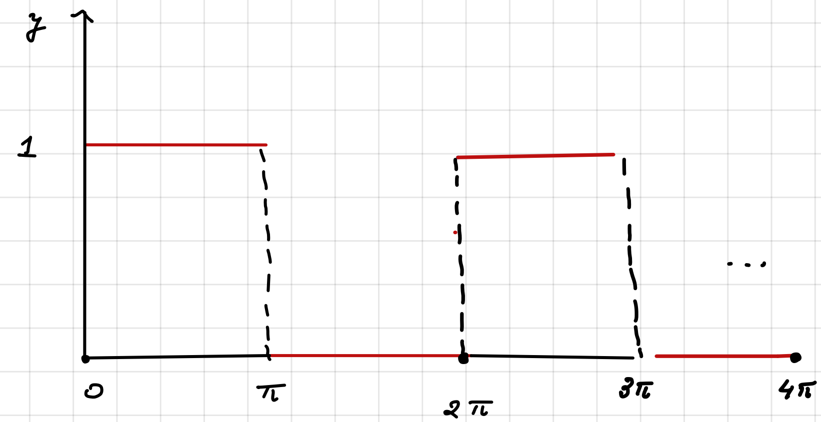
$\boxed{\text{ii}}$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

The proof will be given over the next 2 lectures

Example Let

$$f(x) = \begin{cases} 1, & 0 \le x < \pi \\ 0, & \pi \le x < 2\pi \end{cases}$$



We have seen above that $c_0 = \frac{1}{2}$, $c_n = 0$ for n even, $c_n = \frac{1}{\pi n i}$ for n odd.

We compute $$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2\pi}\cdot \pi = \frac{1}{2}.$$

By Parseval

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$ Substituting, this gives

$$\frac{1}{2} = \underbrace{\frac{1}{4}}_{|c_0|^2} + \sum_{n \text{ odd}} \left|\frac{1}{\pi n i}\right|^2 = \frac{1}{4} + \frac{2}{\pi^2}\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2}$$

The factor " $\frac{2}{\pi^2}$ " accounts for both positive & negative n's.

Thus

$$\frac{1}{2} = \frac{1}{4} + \frac{2}{\pi^2} \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} \implies \boxed{\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} = \frac{\pi^2}{8}}$$

<u>Remark</u> In HWK 7, you will show similar identities e.g.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}} \qquad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$