1.

(i) We apply L'Hopital rule:
\[
\lim_{t \to 0} \frac{f(t) + f(-t) - 2f(0)}{t^2} = \lim_{t \to 0} \frac{f'(t) - f'(-t)}{2t} = \lim_{t \to 0} \frac{f'(t) - f'(0) + f'(-t) - f'(0)}{-2t}.
\]

Each of the fractions above give the limit \( \frac{f''(0)}{2} \) by using the definition of the derivative applied to the function \( f' \). Thus, the final value of the limit is \( \frac{f''(0)}{2} + \frac{f''(0)}{2} = f''(0) \).

(ii) This was part of Midterm 1, Winter 2014, Problem 1. Please refer to the solution posted online.

(iii) By L'Hospital, we have
\[
\lim_{t \to 0} \frac{f(t) - f(0)}{t^2} = \lim_{t \to 0} \frac{f'(t)}{2t}.
\]

Using that \( f'(0) = 0 \) proved in (ii), and the definition of the derivative, we find
\[
\lim_{t \to 0} \frac{f(t) - f(0)}{t^2} = \lim_{t \to 0} \frac{f'(t)}{2t} = \lim_{t \to 0} \frac{f'(t) - f'(0)}{2t} = \frac{f''(0)}{2}.
\]

The above limit can be evaluated via the sequence \( t = \frac{1}{n} \). We obtain
\[
\lim_{n \to \infty} \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n^2}} = \frac{f''(0)}{2}.
\]

Now \( f\left(\frac{1}{n}\right) = 0 \) by assumption for all \( n \geq 1 \). Since \( f \) is differentiable, it is continuous, hence
\[
f(0) = \lim_{n \to \infty} f\left(\frac{1}{n}\right) = 0.
\]

Substituting in the above limit, we find \( f''(0) = 0 \).

2. This problem was Problem 2(ii), Problem Set 3. Please refer to the solutions posted online.

3.

(i) Since \( f \) is increasing, the minimum and maximum of \( f \) over the interval \([\frac{i-1}{n}, \frac{i}{n}]\) is achieved at the endpoints \( \frac{i-1}{n} \) and \( \frac{i}{n} \) respectively. Therefore, we have
\[
U(P_n, f) - L(P_n, f) = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta x_i = \sum_{i=1}^{n} \left( f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right) \cdot \frac{1}{n} = \frac{1}{n} (f(1) - f(0)).
\]

(ii) Pick \( \epsilon > 0 \). Let \( n \) be sufficiently large so that
\[
\frac{f(1) - f(0)}{n} < \epsilon.
\]

Then (i) rewrites as
\[
U(P_n, f) - L(P_n, f) < \epsilon.
\]

By Riemann’s criterion, it follows that \( f \) is integrable.
(iii) We have the string of inequalities
\[
\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{k=1}^n f\left( \frac{k}{n} \right) \right| = \left| \sum_{k=1}^n \int_{k-1/n}^{k/n} f(x) \, dx - \frac{1}{n} \sum_{k=1}^n f\left( \frac{k}{n} \right) \right| \quad \text{(breaking up the integral into pieces)}
\]
\[
= \left| \sum_{k=1}^n \int_{k-1/n}^{k/n} \left( f(x) - f\left( \frac{k}{n} \right) \right) \, dx \right| \quad \text{(brining constants under the integral sign)}
\]
\[
\leq \sum_{k=1}^n \left| \int_{k-1/n}^{k/n} \left( f(x) - f\left( \frac{k}{n} \right) \right) \, dx \right| \quad \text{(using triangle inequality)}
\]
\[
= \sum_{k=1}^n \int_{k-1/n}^{k/n} \left( f\left( \frac{k}{n} \right) - f(x) \right) \, dx \quad \text{(using the function is increasing)}
\]
\[
\leq \sum_{k=1}^n \int_{k-1/n}^{k/n} \left( f\left( \frac{k}{n} \right) - f\left( \frac{k-1}{n} \right) \right) \, dx \quad \text{(using that } f(x) \geq f\left( \frac{k-1}{n} \right)\text{)}
\]
\[
= \sum_{k=1}^n \frac{1}{n} \left( f\left( \frac{k}{n} \right) - f\left( \frac{k-1}{n} \right) \right)
\]
\[
= \frac{1}{n} \cdot (f(1) - f(0)) \leq \frac{1}{n} \quad \text{(as the function takes values in } [0, 1]).
\]

4.

(i) We use the Weierstraß M-test to prove uniform convergence. For } x \in [a, \infty), \text{ we have
\[
\frac{1}{n^2x + 1} < \frac{1}{n^2x} \leq \frac{1}{n^2a}.
\]
Since } \sum_{n=1}^\infty \frac{1}{n^2a} \text{ converges, the series } \sum_{n=1}^\infty \frac{1}{n^2x+1} \text{ converges uniformly over } [a, \infty).

(ii) We prove that } f \text{ is differentiable over } (a, \infty) \text{ for any } a > 0. \text{ This implies that } f \text{ is differentiable at any point } x_0 \in (0, \infty). \text{ Indeed, any such } x_0 \text{ is contained in the interval } (a, \infty) \text{ for } a = \frac{x_0}{2}, \text{ where differentiability will have been already established.}

To prove that } f \text{ is differentiable over } (a, \infty), \text{ we let } s_N \text{ be the partial sum
\[
s_N = \sum_{n=1}^N \frac{1}{n^2x + 1} \implies s'_N = -\sum_{k=1}^N \frac{n^2}{(n^2x + 1)^2}.
\]
The series
\[
\sum_{k=1}^\infty \frac{n^2}{(n^2x + 1)^2}
\]
converges uniformly over } (a, \infty) \text{ by the Weierstraß M-test. Indeed, we have
\[
\frac{n^2}{(n^2x + 1)^2} < \frac{n^2}{(n^2x)^2} = \frac{1}{n^2x^2} < \frac{1}{n^2a^2}.
\]
Since } \sum_{n=1}^\infty \frac{1}{n^2a^2} \text{ converges, it follows that } \sum_{k=1}^\infty \frac{n^2}{(n^2x + 1)^2} \text{ converges uniformly over } (a, \infty). \text{ In other words, } s'_N \text{ converges uniformly over } (a, \infty) \text{ to some function } g. \text{ Since } s_N \text{ converges
uniformly, then by a theorem in class, the limit function $f$ is differentiable over $(a, \infty)$ and in fact $f' = g$.

(iii) Let $s_N$ be the $N^{th}$ partial sum. We have

$$s_{N+1} - s_N = \frac{1}{N^2x + 1}.$$ 

Since for $x_N = \frac{1}{N^2}$, we have

$$(s_{N+1} - s_N)(x_N) = \frac{1}{2},$$

it follows that the supremum norm

$$||s_{N+1} - s_N|| \geq \frac{1}{2}.$$ 

But this violates the Cauchy criterion: should convergence be uniform, for $\epsilon = \frac{1}{3}$, there would exist $N$ such that for $n, m \geq N$, we have

$$||s_n - s_m|| < \frac{1}{3}.$$ 

It suffices to take $n = N, m = N + 1$ to obtain

$$\frac{1}{2} \leq ||s_{N+1} - s_N|| < \frac{1}{3},$$

which is a contradiction.

5.

(i) This was Problem 7, Problem Set 5. Please refer to the solutions posted online.

(ii) By assumption, $0 \leq f(t) \leq 1$. Thus for $x \in [0, 1]$, we have

$$0 \leq F_n(x) = \frac{1}{a_n} \int_0^{a_n x} f(t) \, dt \leq \frac{1}{a_n} \int_0^{a_n x} \, dt = \frac{1}{a_n} \cdot (a_n x) = x \leq 1.$$ 

The sequence $\{F_n\}$ is equicontinuous and uniformly bounded over the compact $[0, 1]$. By Arzelà-Ascoli, it must contain a convergent subsequence.

6.

(i) Let $P(x) = \sum_{i=1}^d c_i x^i$, where $d = \deg f$. If we assume that $P$ is not a constant, then the degree is $d > 0$ and the leading coefficient is $c_d \neq 0$ (otherwise, the degree would be $d - 1$.) We have

$$\lim_{x \to \infty} \frac{P(x)}{x^d} = \sum_{i=0}^d c_i \frac{1}{x^d-i} = c_d \neq 0.$$ 

On the other hand

$$\frac{|P(x)|}{x^d} \leq \frac{M}{|x|^d} \implies \lim_{x \to \infty} \frac{P(x)}{x^d} = 0$$

by the squeeze theorem. The two calculations above give a contradiction. Thus our assumption was wrong, and $P$ must be constant.

(ii) This was Problem 4, Problem Set 6. Please refer to the solution posted online.
7.

(i) By definition, we have
\[ c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} \, dx = \frac{1}{2\pi} \int_0^{2\pi} (\pi - x)e^{-inx} \, dx = \frac{1}{2} \int_0^{2\pi} e^{-inx} \, dx - \frac{1}{2\pi} \int_0^{2\pi} xe^{-inx} \, dx. \]

The first integral is
\[ \int_0^{2\pi} e^{-inx} \, dx = \frac{e^{-inx}}{-in} \bigg|_{x=0}^{x=2\pi} = 0 \]
for \( n \neq 0 \). The second integral is found by integration by parts. We have
\[ \int_0^{2\pi} xe^{-inx} \, dx = \frac{1}{-in} \int_0^{2\pi} x(e^{-inx})' \, dx = \frac{1}{-in} \left( xe^{-inx} \bigg|_{x=0}^{x=2\pi} - \int_0^{2\pi} e^{-inx} \, dx \right) \]
\[ = \frac{1}{-in} \left( 2\pi e^{-2in\pi} - \frac{e^{-inx}}{-in} \bigg|_{x=0}^{x=2\pi} \right) = -\frac{2\pi}{in}. \]
Thus
\[ c_n = \frac{1}{in}. \]

For \( n = 0 \), we need to calculate the coefficient separately. We obtain
\[ c_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) \, dx = 0. \]

(ii) We have
\[ f(x) \sim \sum_n c_n e^{inx} = \sum_{n \neq 0} c_n e^{inx} = \sum_{n \neq 0} \frac{1}{in} e^{inx} = \sum_{n=1}^{\infty} \frac{1}{in} (e^{inx} - e^{-inx}), \]
by collecting the terms corresponding to \( n \) and \(-n\). Since
\[ e^{inx} - e^{-inx} = (\cos nx + i \sin nx) - (\cos nx - i \sin nx) = 2i \sin nx \]
the above calculation shows
\[ f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n} \sin nx. \]

The above identity simply means that the right hand side sine series is the Fourier series of the function \( f(x) \).

However, the problem asks to establish that in fact the function \( f \) equals its Fourier series on the interval \((0, 2\pi)\). This is the issue of establishing the pointwise convergence of the Fourier series. During Winter 2015, we didn’t cover the pointwise convergence of the Fourier series, but the relevant statement is Theorem 8.14 in Rudin.

(iii) Set \( x = \frac{\pi}{2} \) in (ii). We have \( f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \). By (ii) we obtain
\[ \frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin \left(\frac{n\pi}{2}\right) \quad \Rightarrow \quad \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \left(\frac{n\pi}{2}\right). \]
If \( n \) is even, the sine is zero. If \( n = 2k + 1 \) is odd, the sine equals \((-1)^k\), as one can easily check. Thus, the above identity becomes

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.
\]

(iv) By Parseval,

\[
\frac{1}{2\pi} \int_0^{2\pi} f(x)^2 \, dx = \sum_n |c_n|^2 = \sum_{n\neq 0} \left| \frac{1}{in} \right|^2 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

The factor of 2 is obtained because \( n \) and \(-n\) have equal contributions. We compute

\[
\int_0^{2\pi} f(x)^2 \, dx = \int_0^{2\pi} (\pi - x)^2 \, dx = \int_0^{2\pi} (\pi^2 - 2\pi x + x^2) \, dx = \left( \pi^2x - \pi x^2 + \frac{x^3}{3} \right)_{x=0}^{x=2\pi} = \frac{2\pi^3}{3}.
\]

Substituting we obtain

\[
2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi} \int_0^{2\pi} f(x)^2 \, dx = \frac{\pi^2}{3} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

8.

(i) The Fourier coefficients of \( f' \) are found by integration by parts

\[
\frac{1}{2\pi} \int_0^{2\pi} f'(x)e^{-inx} \, dx = \frac{1}{2\pi} \left( f(x)e^{-inx}\big|_0^{2\pi} - \int_0^{2\pi} f(x)(-in)e^{-inx} \, dx \right) = \frac{in}{2\pi} \int_0^{2\pi} f(x)e^{-inx} \, dx = inc_n.
\]

In the above calculations we used the periodicity of \( f \) and \( f' \) to cancel the boundary-terms in the integration by parts formula. By Parseval, we find

\[
\frac{1}{2\pi} \int_0^{2\pi} f'(x)^2 \, dx = \sum_n |inc_n|^2 = \sum_n n^2|c_n|^2.
\]

(ii) Let \( c_n \) be the Fourier coefficients of the function \( f \). By Parseval, we have that

\[
\sum_n |c_n|^2
\]

converges. This will be used below. The proof will have several steps:

- we compute the Fourier coefficients of the function \( f(x + t) - f(x) \)
- we then use Parseval to estimate the integral of \(|f(x + t) - f(x)|^2\)
- we will use theorems on uniform convergence to conclude.

**Step 1.** We compute the Fourier coefficients of the function \( f(x + t) - f(x) \). These equal

\[
\frac{1}{2\pi} \int_0^{2\pi} (f(x + t) - f(x))e^{-inx} \, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x + t)e^{-in(x+t)}e^{int} \, dx - \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} \, dx = e^{int}\cdot \frac{1}{2\pi} \int_t^{2\pi+t} f(y)e^{-iny} \, dy - c_n = c_n(e^{int} - 1).
\]
In the above derivation, we made the change of variables \( y = x + t \), and we also used the periodicity of the function \( f \).

**Step 2.** By Parseval, we have

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(x + t) - f(x)|^2 \, dx = \sum_n |c_n|^2 |e^{int} - 1|^2.
\]

We can rewrite this as follows:

\[
|e^{int} - 1|^2 = |(\cos nt - 1) + i \sin(nt)|^2 = (\cos nt - 1)^2 + (\sin nt)^2 = 2 - 2 \cos nt = 4 \sin^2 \frac{nt}{2}.
\]

Thus,

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(x + t) - f(x)|^2 \, dx = \sum_n |c_n|^2 \sin^2 \frac{nt}{2},
\]

using the half angle formulas. To complete the proof, it suffices to show that

\[
\lim_{t \to 0} \sum_n |c_n|^2 \sin^2 \frac{nt}{2} = 0.
\]

**Step 3.** Consider the series \( \sum_n |c_n|^2 \sin^2 \frac{nt}{2} \) we obtained above. We observed above that \( \sum_n |c_n|^2 \) converges, and furthermore

\[
|c_n|^2 \sin^2 \frac{nt}{2} \leq |c_n|^2.
\]

By the Weierstrass M-test, the series converges uniformly. The partial sums

\[
s_N = \sum_{n=-N}^{N} |c_n|^2 \sin^2 \frac{nt}{2}
\]

are continuous functions in \( t \) and \( s_N(0) = 0 \). The sum

\[
s(t) = \sum_n |c_n|^2 \sin^2 \frac{nt}{2}
\]

is therefore also continuous as uniform limit of continuous functions. Furthermore, by definition,

\[
s(0) = \lim_{N \to \infty} s_N(0) = 0.
\]

Thus by continuity

\[
\lim_{t \to 0} s(t) = 0 \implies \lim_{t \to 0} \sum_n |c_n|^2 \sin^2 \frac{nt}{2} = 0
\]

completing the proof.