Math 140B: Midterm 2
Foundations of Real Analysis

- You have 1 hour and 20 minutes.
- No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- You may not use homework problems (without proof) in your solutions.
1. (10 points) Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = 0$. For every integer $n \geq 1$, let $f_n : [0, \infty) \to [0, \infty)$ and $g_n : [0, \infty) \to [0, \infty)$ be given by

$$f_n(x) = \frac{x}{x + n} \quad \text{and} \quad g_n(x) = \frac{x}{x^2 + n}.$$

(a) (3 points) Prove that $\{f_n\}$ converges uniformly to $f$ on $[0, a]$, for every $a > 0$.

(b) (3 points) Prove that $\{f_n\}$ does not converge uniformly to $f$ on $[0, \infty)$.

(c) (4 points) Prove that $\{g_n\}$ converges uniformly to $f$ on $[0, \infty)$. 
2. (10 points) Consider the following two series

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{x + n^2} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{1}{(x + n^2)^2}, \quad \text{for every } x \geq 0. \]

(a) (5 points) Prove that these series converge uniformly on \([0, \infty)\).
(b) (5 points) Prove that \( f \) is differentiable on \([0, \infty)\) and \( f'(x) + g(x) = 0 \), for every \( x \geq 0 \).
3. (10 points) For every integer $n \geq 1$, let $f_n : [0, 1] \to \mathbb{R}$ be a differentiable function such that $f_n(0) = 0$ and $|f'_n(x)| \leq 1$, for all $x \in [0, 1]$.

(a) (5 points) Prove that the sequence $\{f_n\}$ is equicontinuous.

(b) (5 points) Prove that the sequence $\{f_n\}$ contains a uniformly convergent subsequence.
4. (10 points) Let $f : [-1, 1]$ be a continuous function.
   
   (a) (3 points) Assume that $f(x) = f(-x)$, for all $x \in [-1, 1]$.
   
   Prove that $\int_{-1}^{1} x^{2n+1} f(x) \, dx = 0$, for all integers $n \geq 0$.
   
   (b) (4 points) Assume that $f(x) = f(-x)$, for all $x \in [-1, 1]$.
   
   Prove that there exists a sequence of polynomials $\{P_n\}$ such that the sequence $f_n(x) = P_n(x^2)$ converges uniformly to $f$ on $[-1, 1]$.
   
   (c) (3 points) Assume that $\int_{-1}^{1} x^{2n+1} f(x) \, dx = 0$, for all integers $n \geq 0$.
   
   Prove that $f(x) = f(-x)$, for all $x \in [-1, 1]$.
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