1. We use the supremum norm to test uniform convergence.

(i) We show that \( f_n \to f \). For fixed \( n \), we have

\[
||f_n - f||_{[0,a]} = \sup_{x \in [0,a]} \frac{x}{x + n} = \sup_{x \in [0,a]} \left( 1 - \frac{n}{x + n} \right) = 1 - \frac{n}{a + n} = \frac{a}{a + n}.
\]

Then, since

\[
\lim_{n \to \infty} ||f_n - f|| = \lim_{n \to \infty} \frac{a}{a + n} = 0
\]

we conclude that the \( f_n \)'s converge uniformly to \( f \).

(ii) The same calculation as above shows that for each fixed \( n \) we have

\[
||f_n - f||_{[0,\infty)} = \sup_{x \in [0,\infty)} \left( 1 - \frac{n}{x + n} \right) = 1 - 0 = 1,
\]

which does not converge to 0. Thus \( f_n \) does not converge to \( f \) uniformly over \([0, \infty)\).

(iii) We evaluate

\[
||g_n - f||_{[0,\infty)} = \sup_{x \in [0,\infty)} \frac{x}{x^2 + n}.
\]

We claim that

\[
||g_n||_{[0,\infty)} \leq \frac{1}{2\sqrt{n}}.
\]

This amounts to showing

\[
\frac{x}{x^2 + n} \leq \frac{1}{2\sqrt{n}} \iff 2x\sqrt{n} \leq x^2 + n \iff (x - \sqrt{n})^2 \geq 0,
\]

which is true. Thus, by the squeeze theorem, \( ||g_n - f|| \to 0 \), proving that \( g_n \to f \) over \([0, \infty)\).

2.

(i) To prove uniform convergence we use the Weierstraβ M-test. For the first series, note that

\[
0 < \frac{1}{x + n^2} \leq \frac{1}{n^2}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, we conclude the uniform convergence of \( \sum_{n=1}^{\infty} \frac{1}{x + n^2} \).

For the second series, note that

\[
0 < \frac{1}{(x + n^2)^2} \leq \frac{1}{n^4}.
\]

Since \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges, we conclude the uniform convergence of \( \sum_{n=1}^{\infty} \frac{1}{(x + n^2)^2} \).

(ii) Let \( s_N \) denote the partial sums of the series \( \sum_{n=1}^{\infty} \frac{1}{x + n^2} \). Clearly,

\[
s_N = \sum_{n=1}^{N} \frac{1}{x + n^2} \implies s_N' = \sum_{n=1}^{N} \frac{1}{(x + n^2)^2}.
\]

We showed in (i) that the series \( \sum_{n=1}^{\infty} \frac{1}{(x + n^2)^2} \) converges uniformly to \( g \). Thus, the derivatives of the partial sums

\[
s_N' \to -g.
\]
In addition,

\[ s_N \Rightarrow f. \]

By a theorem proved in class, it follows that the limit of \( s_N \)'s, namely the function \( f \), is differentiable and

\[ f'(x) = -g(x) \implies f'(x) + g(x) = 0. \]

3.

(i) Using the mean value theorem, we conclude that for all \( x, y \) and all \( n \), there exists \( \xi \) such that

\[ |f_n(x) - f_n(y)| = |x - y| \cdot |f'_n(\xi)|. \]

Since \( |f'_n(\xi)| \leq 1 \), we obtain

\[ |f_n(x) - f_n(y)| \leq |x - y|. \]

Fix \( \epsilon > 0 \). For any \( x, y \) such that \( |x - y| < \epsilon \), we obtain that for all \( n \) we have \( |f_n(x) - f_n(y)| < \epsilon \). This reproduces the definition of the equicontinuity of \( f_n \) with \( \delta = \epsilon \).

(ii) From part (i), used for \( y = 0 \), we obtain

\[ |f_n(x)| \leq |x| \leq 1 \]

for all \( x \in [0,1] \). The sequence \( \{f_n\} \) is uniformly bounded. It is also equicontinuous by (i). Thus, by Arzela-Ascoli, the sequence \( \{f_n\} \) must contain a uniformly convergent subsequence.

4.

(i) We have

\[
\int_{-1}^{1} x^{2n+1} f(x) \, dx = \int_{0}^{1} x^{2n+1} f(x) \, dx + \int_{-1}^{0} x^{2n+1} f(x) \, dx = \int_{0}^{1} x^{2n+1} f(x) \, dx + \int_{0}^{1} (-y)^{2n+1} f(-y) \, dy
\]

by using the change of variables \( x = -y \). The above expression rewrites

\[
\int_{0}^{1} x^{2n+1} f(x) \, dx + \int_{0}^{1} (-x)^{2n+1} f(-x) \, dx = \int_{0}^{1} x^{2n+1}(f(x) - f(-x)) \, dx = 0
\]

using that \( f(x) = f(-x) \).

(ii) By Weierstraß approximation, we can find polynomials

\[ Q_n(x) \Rightarrow f(x) \text{ over } [-1,1]. \]

We conclude that

\[ Q_n(-x) \Rightarrow f(-x) = f(x) \]

over the same interval. Adding, we obtain that

\[ \frac{1}{2}(Q_n(x) + Q_n(-x)) \Rightarrow f(x). \]

Let

\[ R_n(x) = \frac{1}{2}(Q_n(x) + Q_n(-x)) \Rightarrow f(x). \]
In the sum $Q_n(x) + Q_n(-x)$ the odd powers of $x$ appear with opposite sign, so they cancel out. Thus, $R_n(x)$ only contains even powers of $x$, or equivalently, it is a polynomial of $x^2$:

$$R_n(x) = P_n(x^2).$$

Therefore that

$$P_n(x^2) \Rightarrow f(x).$$

(iii) Let $g(x) = f(x) - f(-x)$. Since

$$\int_{-1}^{1} f(x)x^{2k+1} \, dx = 0,$$

it follows changing the variables $x \to -x$ that

$$\int_{-1}^{1} f(-x)x^{2k+1} \, dx = 0.$$

Subtracting the last two equations, we obtain

$$\int_{-1}^{1} g(x)x^{2k+1} \, dx = 0.$$

In particular,

$$\int_{-1}^{1} g(x)P_k(x) \, dx = 0$$

for all polynomials $P_k$ containing only odd powers of $k$.

Repeating the argument in part (ii), we can find polynomials

$$P_k \Rightarrow g$$

such that $P_k$ contain only odd powers of $x$. Indeed, it suffices to take $Q_k \Rightarrow f$, and set

$$P_k(x) = Q_k(x) - Q_k(-x) \Rightarrow f(x) - f(-x) = g(x).$$

Then $P_k$ only contain odd powers of $x$ since the even ones cancel out in the expression $Q_k(x) - Q_k(-x)$.

Now $P_k \Rightarrow g$ gives $P_kg \Rightarrow g^2$ since we are dealing with products of uniformly convergent continuous functions defined on compact subsets (see Problem Set 4, Problem 3(iii)). Integrating, we find

$$\int_{-1}^{1} g(x)P_k(x) \, dx \to \int_{-1}^{1} g(x)^2 \, dx.$$

Therefore, since $\int_{-1}^{1} g(x)P_k(x) \, dx = 0$, we obtain

$$\int_{-1}^{1} g(x)^2 \, dx = 0.$$

Since $g(x)^2 \geq 0$, and $g$ is continuous, using Problem Set 3, Problem 1, we find that $g(x) = 0$. Hence, recalling the definition of $g$, we find

$$f(x) - f(-x) = 0$$

for all $x$. 