Problem 1.

Consider $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \frac{nx}{1 + nx^3}.$$

What is the pointwise limit of the sequence $f_n$ as $n \to \infty$? Does the sequence $f_n$ converge uniformly?

Solution: Keeping $x$ fixed, while making $n \to \infty$, we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + nx^3} = \lim_{n \to \infty} \frac{x}{1/n + x^3} = \frac{x}{x^3} = \frac{1}{x^2}.$$

When $x = 0$, the answer above doesn’t make sense, but direct calculation shows that $f_n(0) = 0$. The pointwise limit is the function

$$f : [0, 1] \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{x^2} & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0. \end{cases}$$

The functions $f_n$ are continuous over the interval $[0, 1]$, while the limit function $f$ is not. Therefore, the convergence cannot be uniform.
Problem 2.

Consider the series

\[ f(x) = \sum_{k=1}^{\infty} \frac{1}{1 + k^6 x^2}. \]

(i) Show that the series converges uniformly over any interval \((a, \infty)\) with \(a > 0\).

(ii) Show that \(f\) is differentiable over any interval of the type \((a, \infty)\) with \(a > 0\).

Solution:

(i) We use the Weierstraß \(M\)-test. For \(x \in (a, \infty)\), we estimate

\[ \left| \frac{1}{1 + k^6 x^2} \right| \leq \frac{1}{k^6 a^2} < \frac{1}{k^6 a^2}. \]

Since \(\sum_{k=1}^{\infty} \frac{1}{k^6 a^2} = \frac{1}{a^2} \sum_{k \geq 1} \frac{1}{k^6}\) converges, we conclude that the original series converges uniformly.

(ii) We let

\[ s_N = \sum_{k=1}^{N} \frac{1}{1 + k^6 x^2} \]

denote the partial sums. We have seen above that

\[ s_N \Rightarrow f \]

over the interval \((a, \infty)\). The derivatives of the partial sums are

\[ s'_N = \sum_{k=1}^{N} \frac{-2k^6 x}{(1 + k^6 x^2)^2}. \]

We show that

\[ s'_N \Rightarrow g \]

for some function \(g\) over \((a, \infty)\). It then follows by a theorem in class that \(f\) is differentiable and \(f' = g\). To show uniform convergence of the sequence \(s'_N\), we use the Weierstraß \(M\)-test. We estimate

\[ \left| \frac{-2k^6 x}{(1 + k^6 x^2)^2} \right| \leq \frac{2k^6 x}{(k^6 a^2)^2} = \frac{2}{k^6 a^3} < \frac{2}{k^6 a^3}. \]

To apply the \(M\)-test, it remains to observe that \(\sum_{k=1}^{\infty} \frac{2}{k^6 a^3} \) converges.
Problem 3.

Assume that $f_n : [0, 1] \rightarrow \mathbb{R}$ is a sequence of functions such that $f_n(0) = 0$ and 
\[ |f_n(x) - f_n(y)| \leq |x - y|^{\frac{3}{2}}. \]
Show that $\{f_n\}$ admits a uniformly convergent subsequence.

Solution: We show that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous. Using the Arzelà-Ascoli theorem, it follows that $\{f_n\}$ contains a uniformly convergent subsequence.

To show that $\{f_n\}$ is equicontinuous, let $\epsilon > 0$. Then, for $|x - y| < \epsilon^2$, we have 
\[ |f_n(x) - f_n(y)| \leq |x - y|^{\frac{3}{2}} < \epsilon \]
for all $n$, showing equicontinuity.

To show uniform boundedness, set $y = 0$ and use that $f_n(0) = 0$. For all $n$, the above inequality gives 
\[ |f_n(x)| \leq x^{\frac{1}{2}} \leq 1, \]
completing the proof.
Problem 4.

(i) Let \( A \) be the algebra of functions of the form
\[
g(x) = a_0 + a_1e^x + a_2e^{2x} + \ldots + a_ne^{nx}, \text{ where } a_i \in \mathbb{R}, n \geq 0.
\]
Show that \( A \) separates points and vanishes nowhere.

(ii) Assume that \( f : [0, 1] \to \mathbb{R} \) is a continuous function such that
\[
\int_0^1 f(x)e^{nx} \, dx = 0
\]
holds for all \( n \geq 0 \). Show that \( f = 0 \).

Solution:

(i) Let \( g(x) = e^x \in A \). For points \( x_1, x_2, x_1 \neq x_2 \) we have
\[
g(x_1) \neq g(x_2),
\]
so \( A \) separates points. In addition, for all \( x_1 \), we have
\[
g(x_1) \neq 0,
\]
hence \( A \) vanishes nowhere.

(ii) By Stone-Weierstraß, the algebra \( A \) is dense in the set of continuous functions over the interval \([0,1]\). Thus we can find elements \( g_n \in A \) such that
\[
g_n \rightharpoonup f.
\]
We conclude that
\[
g_nf \rightharpoonup f^2.
\]
In general product of uniformly convergent sequences may not converge uniformly, but over compact sets, this is not an issue by Problem 3, Homework 4. By assumption
\[
\int_0^1 f(x)g_n(x) \, dx = 0
\]
since \( g_n \in A \) is a linear combination of \( e^{kx} \)'s. This uses the linearity of the integral and the fact that \( \int_0^1 f(x)e^{kx} \, dx = 0 \). Now, for uniform convergence, we can integrate to find that
\[
\int_0^1 f(x)g_n(x) \, dx \to \int_0^1 f(x)^2 \, dx.
\]
We conclude that
\[
\int_0^1 f(x)^2 \, dx = 0.
\]
Since \( f(x)^2 \geq 0 \) and \( f^2 \) is continuous, Homework 3, Problem 1 implies that \( f = 0 \).