Math 145 Solution Set 3

Problem 1. *Find the irreducible components of the affine algebraic set $x^2 - yz = xz - x = 0$ in $\mathbb{A}^3$.*

Answer: Let X be the affine algebraic set given by the two equations above. The second equation can be written as

$$xz - x = x(z-1) = 0 \implies x = 0 \text{ or } z = 1.$$

Case 1. If $x = 0$, the first equation becomes

$$x^2 - yz = yz = 0 \implies y = 0 \text{ or } z = 0.$$

Therefore, there are two possibilities:

$$x = y = 0 \text{ or } x = z = 0\,.$$

These correspond to two algebraic sets $X_1 =$ the z-axis and $X_2 =$ the y-axis, defined by the equations

$$X_1 = \mathcal{Z}(y, z) \text{ and } X_2 = \mathcal{Z}(x, z).$$

Case 2. If $z = 1$, the first equation becomes

$$x^2 - yz = x^2 - y = 0.$$

This zero set is a parabola which can be rewritten as

$$X_3 = \mathcal{Z}(z - 1, x^2 - y).$$

According to discussion above,

$$X = X_1 \cup X_2 \cup X_3.$$

We claim that the X_i's are irreducible.

Indeed, X_1 is irreducible if and only if (y, z) is a prime ideal in $\mathbb{C}[x, y, z]$, which is equivalent to $\mathbb{C}[x, y, z]/(y, z)$ is an integral domain. This is the case since

$$\mathbb{C}[x, y, z]/(y, z) \cong \mathbb{C}[x]\,,$$

is surely an integral domain. By the same argument, X_2 is also irreducible.

Finally, $X_3 = \mathcal{Z}(z - 1, x^2 - y)$ is irreducible. Indeed $(z - 1, x^2 - y)$ is a prime ideal in $\mathbb{C}[x, y, z]$. This is the case since

$$\mathbb{C}[x, y, z]/(z - 1, x^2 - y) \cong \mathbb{C}[x, y]/(x^2 - y) \cong \mathbb{C}[x]\,,$$

is an integral domain.

This shows X_1, X_2, X_3 are irreducible components of the affine algebraic set X. □

Problem 2. *Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in $\mathbb{A}^3$.*

Answer: If $x = 0$, then the two equations imply that $y = z = 0$. Similarly, if $y = 0$, then $x = z = 0$. Let us assume that $x \neq 0$, $y \neq 0$.

Write the first equation as

$$\frac{y}{x} = \frac{z}{y} := t.$$

Thus,

$$y = tx,\ z = t^2 x.$$

The second equation becomes

$$z^3 = x^5 \iff t^6x^3 = x^5 \iff x^2 = t^6 \iff x = \pm t^3.$$

This implies

$$y = \pm t^4, z = \pm t^5.$$

Thus $(x, y, z) = (t^3, t^4, t^5)$ or $(-t^3, -t^4, -t^5)$ for some $t \in \mathbb{C}$.

Letting

$$X_1 = \{(t^3, t^4, t^5) |\, t \in \mathbb{C}\} \text{ and } X_2 = \{(-t^3, -t^4, -t^5) |\, t \in \mathbb{C}\}$$

we obtain

$$X = X_1 \cup X_2.$$

We claim that X_1, X_2 are the irreducible components of X.

First, there are polynomial maps

$$f_1 : \mathbb{A}^1 \to \mathbb{A}^3 \quad t \to (t^3, t^4, t^5) \text{ and } f_2 : \mathbb{A}^1 \to \mathbb{A}^3 \quad t \to (-t^3, -t^4, -t^5)\,.$$

Since $X_1 = f_1(\mathbb{A}^1)$, $X_2 = f_2(\mathbb{A}^1)$ and $\mathbb{A}^1$ is irreducible, the images X_1 and X_2 are irreducible.

It remains to explain that X_1 and X_2 are algebraic sets. We claim

$$X_1 = \mathcal{Z}(y^3 - x^4, z^3 - x^5, z^4 - y^5),\ X_2 = \mathcal{Z}(y^3 + x^4, z^3 - x^5, z^4 + y^5).$$

Let us check this claim for X_1 only. This follows using the same strategy as before. Indeed, if $x = 0$ then $y = z = 0$, so we may assume $x \neq 0$. Let

$$t = \frac{y}{x}.$$

From the first equation we have

$$x = \left(\frac{y}{x}\right)^3 = t^3 \implies y = t^4.$$

Dividing the last two equations we get

$$z = \frac{y^5}{x^5} = t^5 \implies (x, y, z) = (t^3, t^4, t^5),$$

as claimed. The verification for X_2 is entirely similar.

□

Problem 3. *Prove that the irreducible components of a Noetherian topological space are unique. That is, if*

$$X = \cup X_i = \cup Y_i$$

such that all sets X_i and Y_i are irreducible, assumed to be irredundant (e.g. $X_i \not\subset X_j$ and similarly $Y_i \not\subset Y_j$) then X_i's are a permutation of the Y_i's.

Answer: Fix an index j. We have

$$X_j = (\bigcup Y_i) \cap X_j = \bigcup (Y_i \cap X_j)\,.$$

For each i, $Y_i \cap X_j$ is a closed subset of X_j. Thus X_j is the union of closed sets $(Y_i \cap X_j)$. Since X_j is irreducible, there exists i such that

$$Y_i \cap X_j = X_j \implies X_j \subset Y_i.$$

Therefore, there exists an assignment

$$f : j \mapsto i = f(j) \text{ such that } X_j \subset Y_{f(j)}.$$

Similarly, there exists an assignment g such that

$$Y_j \subset X_{g(j)}$$

for all j.

Combining, we get

$$X_j \subset Y_{f(j)} \subset X_{g(f(j))}\,.$$

Because the sets X_i are irredundant, we must have

$$g(f(j)) = j$$

for every j. Similarly

$$Y_j \subset X_{g(j)} \subset X_{f(g(j)} \implies f(g(j)) = j.$$

Therefore, f and g are inverse bijections, and the X_i's are a permutation of the Y_i's. □

Problem 4. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be distinct algebraic subsets of $\mathbb{A}^2$ given as $\mathcal{Z}_1 = \mathcal{Z}(f)$ and $\mathcal{Z}_2 = \mathcal{Z}(g)$ where $f, g \in k[X, Y]$ are irreducible polynomials. Show that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ intersect in finitely many points, or equivalently that f and g have finitely many common zeros:*

(i) *Consider the resultant $R_{f,g} \in k[X]$, viewing f and g as polynomials in Y with coefficients in $k[X]$. Explain why the resultant cannot be the zero polynomial.*

(ii) *Note that if $(a, b) \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, then $f(a, Y)$ and $g(a, Y)$ have a common factor $Y - b$. Conclude that a must be a root of $R_{f,g}$. This shows that a can only take on only finitely many values. Conclude the same about b, and complete the proof.*

Answer: (i) We compute $R_{f,g}$ by viewing f and g as polynomials in Y with coefficients in $A = k[X]$ and take determinant. If $R_{f,g}$ is the zero polynomial, then since the ring of coefficients $A = k[X]$ is an UFD, we conclude that f and g have a non-constant common factor h. By hypothesis $f, g \in k[X, Y]$ are irreducible polynomials, hence $f = g$. This contradicts to the fact $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are distinct algebraic subsets. Therefore $R_{f,g}$ is not the zero polynomial.

(ii) If $(a, b) \in \mathcal{Z}_1 \cap \mathcal{Z}_2$, then $f(a, b) = g(a, b) = 0$. Therefore the polynomials $f(a, Y)$ and $g(a, Y)$ have a common root b, so $f(a, Y)$ and $g(a, Y)$ have a common factor $Y - b$. Therefore

$$R_{f,g}(a) = 0.$$

By (i), $R_{f,g}(X) \in \mathbb{C}[x]$ is not the zero polynomial so it only has finite roots. So there are only finite choices of a. By symmetry, there are only finite choices of b. Thus f and g only have finitely many common zeros.

□

Problem 5. *An algebraic set $\mathcal{Z} \subset \mathbb{A}^2$ defined by an irreducible polynomial F of degree 2 is called an irreducible conic. Show that any irreducible conic can be written in the form*

$$Y - X^2 = 0 \;\; or \;\; XY - 1 = 0$$

after an affine change of coordinates in $\mathbb{A}^2$.

Answer: Let

$$F(x, y) = ax^2 + by^2 + cxy + dx + ey + f.$$

We discuss the following cases.

Case 1. If $a = b = 0$, then

$$F(x,y) = cxy + dx + ey + f = c\left(xy + \frac{d}{c}x + \frac{e}{c}y\right) + f = c\left(x + \frac{e}{c}\right)\left(y + \frac{d}{c}\right) + \tilde{f}$$

If $\tilde{f} = 0$, F is reducible, which is not allowed. Therefore $\tilde{f} \neq 0$. Let

$$X = -\frac{c}{\tilde{f}}\left(x + \frac{e}{c}\right) \text{ and } Y = y + \frac{d}{c}\,, \text{ so that}$$

$$F = -\tilde{f}XY + \tilde{f}\,.$$

Thus $F(x,y) = 0$ implies

$$XY - 1 = 0.$$

After an affine change of coordinates, the conic $\mathcal{Z}(F)$ becomes

$$XY - 1 = 0.$$

Case 2. If either a or b is not 0, without loss of generality, we may assume $a \neq 0$. Then,

$$F = ax^2 + by^2 + cxy + dx + ey + f = a\left(x + \frac{c}{2a}y\right)^2 + \tilde{b}y^2 + dx + ey + f$$

Let

$$x_1 = \sqrt{a}\left(x + \frac{c}{2a}y\right) \text{ and } y_1 = y\,,$$

(choose any one of the square roots). There exist constants $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ such that

$$F = x_1^2 + by_1^2 + \tilde{d}x_1 + \tilde{e}y_1 + \tilde{f} = \left(x_1 + \frac{\tilde{d}}{2}\right)^2 + by_1^2 + \tilde{e}y_1 + \tilde{\tilde{f}}.$$

Let

$$x_2 = x_1 + \frac{\tilde{d}}{2}.$$

Subcase (i): If $b = 0$,

$$F = x_2^2 + \tilde{e}y_1 + \tilde{\tilde{f}}\,.$$

We claim $\tilde{e} \neq 0$ because otherwise

$$F = (x_2 + i\sqrt{\tilde{\tilde{f}}})(x_2 - i\sqrt{\tilde{\tilde{f}}})$$

is reducible. Let

$$X = x_2,\ Y = -(\tilde{e}y_1 + \tilde{\tilde{f}}).$$

Then

$$F = X^2 - Y.$$

Subcase (ii): If $b \neq 0$, let

$$y_2 = \sqrt{b}y_1 + \frac{\tilde{e}}{2\sqrt{b}}$$

so that

$$F = x_2^2 + y_2^2 + g.$$

Letting

$$X = \sqrt{-g}(x_2 + iy_2) \text{ and } Y = \sqrt{-g}(x_2 - iy_2),$$

we have

$$F = -g(XY - 1).$$

Therefore, the conic $\mathcal{Z}(F)$ can be written in the form

$$XY - 1 = 0$$

after an affine change of coordinates.

□

Problem 6. *(Baby Bezout's Theorem) Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using the previous problem, show that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ intersect in at most 4 points.*

Answer: Suppose $\mathcal{Z}_1 = \mathcal{Z}(f)$ and $\mathcal{Z}_2 = \mathcal{Z}(g)$ for some polynomials $f, g \in \mathbb{C}[X, Y]$. Using the previous problem, we can change coordinates to achieve

$$f = XY - 1 \text{ or } f = Y - X^2.$$

We have no control over g, so let us write

$$g = aX^2 + bY^2 + cXY + dX + eY + f\,.$$

Let (x, y) be a point in $\mathbb{A}^2$ such that

$$f(x, y) = g(x, y) = 0,$$

i.e. $(x, y) \in \mathcal{Z}_1 \cap \mathcal{Z}_2$.

Case 1. If $f = XY - 1$, then

$$y = 1/x \text{ and } x \neq 0.$$

Substituting into g, we get

$$0 = g(x, 1/x) = ax^2 + \frac{b}{x^2} + cx \cdot \frac{1}{x} + dx + \frac{e}{x} + f\,.$$

Multiplying by x^2, we obtain

$$ax^4 + b + cx^2 + dx^3 + ex + fx^2 = 0\,.$$

This degree 4 polynomial has at most 4 roots. Therefore, $\mathcal{Z}_1 \cap \mathcal{Z}_2$ contains at most 4 points.

Case 2. If $f = Y - X^2$, then

$$y = x^2.$$

Substituting into g, we get

$$0 = g(x, x^2) = ax^2 + bx^4 + cx^3 + dx + ex^2 + f.$$

Thus x satisfies a polynomial of degree at most 4, so there are at most 4 values for x. As $y = x^2$, it follows that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ contains at most 4 points.

□