Math 145, Problem Set 3. Due Friday, April 25.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. Find the irreducible components of the affine algebraic set $x^2 - yz = xz - x = 0$ in $\mathbb{A}^3$.

2. Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in $\mathbb{A}^3$.

3. Prove that the irreducible components of a Noetherian topological space are unique. That is, if $X = \bigcup X_i = \bigcup Y_i$ such that all sets $X_i$ and $Y_i$ are irreducible, assumed to be irredundant (e.g. $X_i \not\subset X_j$ and similarly $Y_i \not\subset Y_j$ for $i \neq j$) then $X_i$’s are a permutation of the $Y_i$’s.

4. Let $Z_1$ and $Z_2$ be distinct algebraic subsets of $\mathbb{A}^2$ given as $Z_1 = Z(f)$ and $Z_2 = Z(g)$ where $f, g \in k[X,Y]$ are irreducible polynomials. Show that $Z_1 \cap Z_2$ intersect in finitely many points, or equivalently that $f$ and $g$ have finitely many common zeros:

   (i) Consider the resultant $R_{f,g} \in k[X]$, viewing $f$ and $g$ as polynomials in $Y$ with coefficients in $k[X]$. Explain why the resultant cannot be the zero polynomial.

   (ii) Note that if $(a, b) \in Z_1 \cap Z_2$, then $f(a,Y)$ and $g(a,Y)$ have a common factor $Y - b$. Conclude that $a$ must be a root of $R_{f,g}$. This shows that $a$ can only take on only finitely many values. Conclude the same about $b$, and complete the proof.

5. An algebraic set $Z \subset \mathbb{A}^2$ defined by an irreducible polynomial $f$ of degree 2 is called an irreducible conic. Show that any irreducible conic can be written in the form

   \[ Y - X^2 = 0 \text{ or } XY - 1 = 0 \]

   after an affine change of coordinates in $\mathbb{A}^2$.

   Remark: An affine change of coordinates taking $(x, y)$ into $(X, Y)$ is a transformation of the form

   \[
   \begin{pmatrix}
   X \\
   Y
   \end{pmatrix} = A \begin{pmatrix}
   x \\
   y
   \end{pmatrix} + b,
   \]

   where $A$ is a $2 \times 2$ invertible matrix and $b \in \mathbb{A}^2$ is a vector.

   For instance, the conic

   \[ x^2 + y^2 = 1 \]

   becomes $XY - 1 = 0$ after the change of coordinates

   \[ X = x + \sqrt{-1}y, Y = x - \sqrt{-1}y. \]

   Hint: Let $f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$. If $a = b = 0$, show that the conic can be written as $XY = 1$ after changing coordinates. Otherwise, assume that $a \neq 0$ and complete the square in $ax^2 + cxy$. In suitable coordinates the polynomial $f$ changes to $X^2 + b'Y^2 + d'X + e'Y + f'$. Continue in this fashion, completing the squares a few more times if necessary.

6. (Baby Bezout’s Theorem) Let $Z_1$ and $Z_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using the previous problem, show that $Z_1$ and $Z_2$ intersect in at most 4 points.