Math 145, Problem Set 4. Due Friday, May 2.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

1. Let \( f \) and \( g \) be distinct irreducible polynomials in \( k[X,Y] \) of degrees \( d \) and \( e \). Show that \( \mathcal{Z}(f) \) and \( \mathcal{Z}(g) \) intersect in at most \( de \) points:
   (i) Let \( S \) be the set of intersection points. We proved in the last problem set that \( S \) is finite. Let \( n \) be the number of points in \( S \). We would like to show that \( n \leq de \).
   To begin, write \( S = \{(a_1,b_1),\ldots,(a_n,b_n)\} \).
   Explain why there exists \( \lambda \in k \) such that \( \lambda a_i + b_i \neq \lambda a_j + b_j \), for \( i \neq j \).
   (ii) Let \( F(X,Y) = f(X,Y - \lambda X), G(X,Y) = g(X,Y - \lambda X) \).
   Show that \( F \) and \( G \) have degrees at most \( d \) and \( e \). Show that \( F \) and \( G \) are distinct irreducible polynomials. Show that \( \mathcal{Z}(F) \cap \mathcal{Z}(G) \) consists in \( n \) points \( \{(a_1,c_1),\ldots,(a_n,c_n)\} \)
   where \( c_i \neq c_j \) for all \( i \neq j \).
   (iii) Writing \( F(X,Y) = F_d(Y)X^d + \ldots + F_0(Y), G(X,Y) = G_e(Y)X^e + \ldots + G_0(Y) \)
   as polynomials in \( X \) with coefficients in \( k[Y] \), observe that
   \( \deg F_i \leq d - i \) for all \( 0 \leq i \leq d \), and \( \deg G_j \leq e - j \), for all \( 0 \leq j \leq e \).
   Prove that the resultant \( R_{F,G} \in k[Y] \) has degree at most \( de \), hence it has at most \( de \) roots.
   (iv) Conclude that if \( (a,c) \in \mathcal{Z}(F) \cap \mathcal{Z}(G) \), then \( c \) can take on at most \( de \) values. Conclude that \( n \leq de \).

2. Which of the following algebraic sets are isomorphic:
   (i) \( \mathbb{A}^1 \)
   (ii) \( \mathcal{Z}(xy) \subset \mathbb{A}^2 \)
   (iii) \( \mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2 \)
   (iv) \( \mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2 \)
   (v) \( \mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2 \).

3. Consider the cubic curve \( y^2 = x(x-1)(x-\lambda) \) in \( \mathbb{A}^2 \), where \( \lambda \neq 0,1 \). Show that there are no nonconstant morphisms \( f : \mathbb{A}^1 \to X \).
*Hint:* Write \( f(t) = (g(t), h(t)) \) and observe that

\[
h(t)^2 = g(t)(g(t) - 1)(g(t) - \lambda)
\]

must be a perfect square.

4. Show that \( X = \mathbb{A}^2 \setminus \{(0, 0)\} \) cannot be isomorphic to an affine algebraic set:

(i) Show that if \( f : X \to \mathbb{A}^1 \) is a regular function on \( X \), then \( f \) must be a polynomial in \( k[x, y] \). To see this, write \( f \) as a quotient \( g/h \) of two polynomials \( g, h \) without common factors. Observe that \( h \) cannot vanish on \( \mathbb{A}^2 \setminus \{(0, 0)\} \), and conclude that \( h \) must be constant.

(ii) If \( \Phi : Y \to X \) is an isomorphism between an affine algebraic set \( Y \) and \( X \), consider the composition \( \Psi = \iota \circ \Phi : Y \to \mathbb{A}^2 \), where \( \iota : X \to \mathbb{A}^2 \) is the inclusion. Show that the morphism \( \Psi \) induces an isomorphism on coordinate rings. Conclude that \( \Psi \) must be an isomorphism, hence \( \iota \) must be an isomorphism, which must be a contradiction.

5. From the textbook, solve 4.8 and 4.10, page 77. Solve 5.1, page 90.