Math 145, Problem Set 5. Due Friday, May 16.

You may assume that the ground field is $k = \mathbb{C}$.

1. (*Products of affine varieties.*) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine algebraic sets.

(i) Show that $X \times Y \subset \mathbb{A}^{n+m}$ is also an affine algebraic set.

(ii) *Extra credit and entirely optional.* Show that if X and Y are irreducible, $X \times Y$ is also irreducible. Look up problem 5.10 (i) in the textbook for a hint.

2. (*Intersections with hypersurfaces.*)

(i) Let $H \subset \mathbb{P}^n$ be a linear hyperplane in $\mathbb{P}^n$ (e.g. H is cut out by one linear homogeneous equation in $\mathbb{P}^n$). Show that the *quasiprojective* algebraic set $\mathbb{P}^n \setminus H$ is isomorphic to $\mathbb{A}^n$.

Hint: Explain first that after changing coordinates you may assume that H is cut out by the equation

$$H = \{X_n = 0\}.$$

Show that

$$\pi : \mathbb{P}^n \setminus H \to \mathbb{A}^n, [X_0 : \ldots : X_n] \to \left(\frac{X_0}{X_n}, \ldots, \frac{X_{n-1}}{X_n}\right)$$

is an isomorphism.

(ii) Show that if X is a degree d hypersurface in $\mathbb{P}^n$ then $\mathbb{P}^n \setminus X$ admits a nonconstant morphism to an affine space. (In fact, it's not much harder to show that $\mathbb{P}^n \setminus X$ is isomorphic to an affine variety).

Hint: You may want to use the d-fold Veronese embedding of $\mathbb{P}^n$ to reduce to part (i). Note that the equation of X becomes linear in the Veronese coordinates.

(iii) Conclude that if X is a hypersurface in $\mathbb{P}^n$ and $Y \subset \mathbb{P}^n$ is a projective variety which is not a point, then the intersection of X and Y is non-emtpy.

(iv) If X is a degree d hypersurface in $\mathbb{P}^n$ and L is a line in $\mathbb{P}^n$ not contained in X, show that L and X intersect in at most d points. In fact, they intersect in exactly d points counted with multiplicity.

Hint: After a change of coordinates, you may assume that L is cut out by the equations

$$X_1 = \ldots = X_{n-2} = 0.$$

3. (*Rational varieties.*) Two projective varieties X and Y are birational if there are rational maps

$$f : X \dashrightarrow Y \ , \ g : Y \dashrightarrow X.$$

which are rational inverses to each other. We say that X is rational if X is birational to $\mathbb{P}^n$ for some n.

(i) Show that a birational isomorphism $f : X \dashrightarrow Y$ induces an isomorphism of the fields of rational functions $f^\star : K(Y) \to K(X)$. The converse is also true but you don't have to prove it.

Remark: It follows that if X is rational, then $K(X) \cong K(\mathbb{P}^n) \cong k(t_1, \ldots, t_n)$, the field of rational fractions in the variables $t_1, \ldots, t_n$. (To see the last isomorphism, elements in $K(\mathbb{P}^n)$ are fractions $f(t_0 : \ldots : t_n)/g(t_0 : \ldots : t_n)$, with f and g homogeneous of the same degree. The isomorphism with $k(t_1, \ldots, t_n)$ is obtained setting the variable $t_0 = 1$. This destroys the homogeneity of the numerator and denominator.)

(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if X and Y are rational, then $X \times Y$ is rational.

Remark: It is very difficult to determine if a given variety is rational. We have seen that lines and conics in $\mathbb{P}^2$ are rational, while elliptic curves in $\mathbb{P}^2$ are not. Twisted cubics in $\mathbb{P}^3$ are rational. We will prove below that quadrics in $\mathbb{P}^3$ are rational. Cubic surfaces in $\mathbb{P}^3$ also turn out to be rational. However, most varieties are not rational.

4. (*Quadrics are rational.*) A quadric $Q \subset \mathbb{P}^n$ is non-degenerate if it is not contained in a linear hyperplane of $\mathbb{P}^n$. For obvious reasons, we will only consider non-degenerate quadrics.

(i) Show that a non-degenerate irreducible quadric Q in $\mathbb{P}^3$ can be written in the form

$$xy = zw$$

after a suitable change of homogeneous coordinates. Combining this result with the Segre embedding, conclude that any quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, hence it is rational.

Hint: Using the argument we gave in class for conics in $\mathbb{P}^2$, show first that the quadric can be brought into the form $x^2 + y^2 + z^2 + w^2 = 0$.

(ii) In general, show that any non-degenerate irreducible quadric $Q \subset \mathbb{P}^{n+1}$ is birational to $\mathbb{P}^n$.

Hint: Pick $p \in Q$ and assume after a change of coordinates that $p = [1 : 0 : \ldots : 0]$. Consider the linear hyperplane

$$H = \{X_0 = 0\}.$$

For any $q \in Q$, define $f(q)$ to be the point of intersection of the line pq with H. Show that

$$f : Q \dashrightarrow H$$

is a birational isomorphism. This rational map is called the linear projection from p.

5. *(Quadrics and curves.)*

(i) Show that two quadrics in $\mathbb{P}^3$ intersect in an elliptic curve. More precisely, consider the following two quadrics in $\mathbb{P}^3$:

$$Q_1 = \mathcal{Z}(xw - yz), Q_2 = (yw - (x - z)(x - \lambda z)).$$

Show that the intersection of the two quadrics is isomorphic to the elliptic curve $\overline{E}_\lambda$.

(ii) Show that the twisted cubic in $\mathbb{P}^3$ is the intersections of the three quadrics

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xt - yz), Q_3 = \mathcal{Z}(yt - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic.

6. (*Introduction to moduli theory.*) Show that for any 3 lines L_1, L_2, L_3 in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric Q and not on the polynomial defining it. Let us denote this point by p_Q. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

Remark: The projective space $\mathbb{P}^9$ is said to be the moduli space of quadrics in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subpace

$$H_L \subset \mathbb{P}^9$$

such that

$$L \subset Q \text{ iff and only if } p_Q \in H_L.$$

Hint: You may want to change coordinates to assume that the line L is cut out by the equations $X_0 = X_1 = 0$. This will force 3 of the coefficients of Q to be zero. Which ones? What is the subspace H_L in this case?

(iii) Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that L_1, L_2, L_3 are contained in a quadric Q.

(iv) Explain (briefly) that if L_1, L_2, L_3 are disjoint lines, then Q can be assumed to be irreducible.

Hint: What are the reducible quadrics?