Math 145, Problem Set 5. Due Friday, May 16.

You may assume that the ground field is \( k = \mathbb{C} \).

1. **(Products of affine varieties.)** Let \( X \subset \mathbb{A}^n \) and \( Y \subset \mathbb{A}^m \) be affine algebraic sets.

   (i) Show that \( X \times Y \subset \mathbb{A}^{n+m} \) is also an affine algebraic set.

   (ii) *Extra credit and entirely optional.* Show that if \( X \) and \( Y \) are irreducible, \( X \times Y \) is also irreducible. Look up problem 5.10 (i) in the textbook for a hint.

2. **(Intersections with hypersurfaces.)**

   (i) Let \( H \subset \mathbb{P}^n \) be a linear hyperplane in \( \mathbb{P}^n \) (e.g. \( H \) is cut out by one linear homogeneous equation in \( \mathbb{P}^n \)). Show that the quasiprojective algebraic set \( \mathbb{P}^n \setminus H \) is isomorphic to \( \mathbb{A}^n \).

   *Hint:* Explain first that after changing coordinates you may assume that \( H \) is cut out by the equation \( H = \{ X_n = 0 \} \).

   Show that
   \[
   \pi : \mathbb{P}^n \setminus H \to \mathbb{A}^n, [X_0 : \ldots : X_n] \to \left( \frac{X_0}{X_n}, \ldots, \frac{X_{n-1}}{X_n} \right)
   \]
   is an isomorphism.

   (ii) Show that if \( X \) is a degree \( d \) hypersurface in \( \mathbb{P}^n \) then \( \mathbb{P}^n \setminus X \) admits a nonconstant morphism to an affine space. (In fact, it’s not much harder to show that \( \mathbb{P}^n \setminus X \) is isomorphic to an affine variety).

   *Hint:* You may want to use the \( d \)-fold Veronese embedding of \( \mathbb{P}^n \) to reduce to part (i). Note that the equation of \( X \) becomes linear in the Veronese coordinates.

   (iii) Conclude that if \( X \) is a hypersurface in \( \mathbb{P}^n \) and \( Y \subset \mathbb{P}^n \) is a projective variety which is not a point, then the intersection of \( X \) and \( Y \) is non-empty.

   (iv) If \( X \) is a degree \( d \) hypersurface in \( \mathbb{P}^n \) and \( L \) is a line in \( \mathbb{P}^n \) not contained in \( X \), show that \( L \) and \( X \) intersect in at most \( d \) points. In fact, they intersect in exactly \( d \) points counted with multiplicity.

   *Hint:* After a change of coordinates, you may assume that \( L \) is cut out by the equations
   \[
   X_1 = \ldots = X_{n-2} = 0.
   \]

3. **(Rational varieties.)** Two projective varieties \( X \) and \( Y \) are birational if there are rational maps
   \[
   f : X \dashrightarrow Y, \quad g : Y \dashrightarrow X,
   \]
   which are rational inverses to each other. We say that \( X \) is rational if \( X \) is birational to \( \mathbb{P}^n \) for some \( n \).

   (i) Show that a birational isomorphism \( f : X \dashrightarrow Y \) induces an isomorphism of the fields of rational functions \( f^* : K(Y) \to K(X) \). The converse is also true but you don’t have to prove it.

   *Remark:* It follows that if \( X \) is rational, then \( K(X) \cong K(\mathbb{P}^n) \cong k(t_1, \ldots, t_n) \), the field of rational fractions in the variables \( t_1, \ldots, t_n \). (To see the last isomorphism, elements in \( K(\mathbb{P}^n) \) are fractions \( f(t_0 : \ldots : t_n)/g(t_0 : \ldots : t_n) \), with \( f \) and \( g \) homogeneous of the same degree. The isomorphism with \( k(t_1, \ldots, t_n) \) is obtained setting the variable \( t_0 = 1 \). This destroys the homogeneity of the numerator and denominator.)
(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X$ and $Y$ are rational, then $X \times Y$ is rational.

Remark: It is very difficult to determine if a given variety is rational. We have seen that lines and conics in $\mathbb{P}^2$ are rational, while elliptic curves in $\mathbb{P}^2$ are not. Twisted cubics in $\mathbb{P}^3$ are rational. We will prove below that quadrics in $\mathbb{P}^3$ are rational. Cubic surfaces in $\mathbb{P}^3$ also turn out to be rational. However, most varieties are not rational.

4. (Quadrics are rational.) A quadric $Q \subset \mathbb{P}^n$ is non-degenerate if it is not contained in a linear hyperplane of $\mathbb{P}^n$. For obvious reasons, we will only consider non-degenerate quadrics.

(i) Show that a non-degenerate irreducible quadric $Q$ in $\mathbb{P}^3$ can be written in the form $xy = zw$ after a suitable change of homogeneous coordinates. Combining this result with the Segre embedding, conclude that any quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, hence it is rational.

Hint: Using the argument we gave in class for conics in $\mathbb{P}^2$, show first that the quadric can be brought into the form $x^2 + y^2 + z^2 + w^2 = 0$.

(ii) In general, show that any non-degenerate irreducible quadric $Q \subset \mathbb{P}^{n+1}$ is birational to $\mathbb{P}^n$.

Hint: Pick $p \in Q$ and assume after a change of coordinates that $p = [1 : 0 : \ldots : 0]$. Consider the linear hyperplane $H = \{X_0 = 0\}$.

For any $q \in Q$, define $f(q)$ to be the point of intersection of the line $pq$ with $H$. Show that $f : Q \dasharrow H$ is a birational isomorphism. This rational map is called the linear projection from $p$.

5. (Quadrics and curves.)

(i) Show that two quadrics in $\mathbb{P}^3$ intersect in an elliptic curve. More precisely, consider the following two quadrics in $\mathbb{P}^3$:

$Q_1 = Z(xw - yz), Q_2 = (yw - (x - z)(x - \lambda z)).$

Show that the intersection of the two quadrics is isomorphic to the elliptic curve $E_\lambda$.

(ii) Show that the twisted cubic in $\mathbb{P}^3$ is the intersections of the three quadrics $Q_1 = Z(xz - y^2), Q_2 = Z(xt - yz), Q_3 = Z(yt - z^2)$.

Show that any two of these quadrics will not intersect in the twisted cubic.

6. (Introduction to moduli theory.) Show that for any 3 lines $L_1, L_2, L_3$ in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric $Q$ and not on the polynomial defining it. Let us denote this point by $p_Q$. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

Remark: The projective space $\mathbb{P}^9$ is said to be the moduli space of quadrics in $\mathbb{P}^3$. 

(ii) Consider a line \( L \subset \mathbb{P}^3 \). Show that there is a codimension 3 projective linear subspace
\[ H_L \subset \mathbb{P}^9 \]
such that
\[ L \subset Q \iff p_Q \in H_L. \]

\textit{Hint:} You may want to change coordinates to assume that the line \( L \) is cut out by the equations \( X_0 = X_1 = 0 \). This will force 3 of the coefficients of \( Q \) to be zero. Which ones? What is the subspace \( H_L \) in this case?

(iii) Show that any three codimension 3 projective linear subspaces of \( \mathbb{P}^9 \) intersect. In particular, show that
\[ H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset, \]
and conclude that \( L_1, L_2, L_3 \) are contained in a quadric \( Q \).

(iv) Explain (briefly) that if \( L_1, L_2, L_3 \) are disjoint lines, then \( Q \) can be assumed to be irreducible.

\textit{Hint:} What are the reducible quadrics?