

Math 145, Midterm Exam, Due May 7 in class.

You may assume that the ground field is $k = \mathbb{C}$.

Problem 1. (10 points.) Let $n \geq 2$, and let $S = \{a_1, \dots, a_n\}$ be a finite set with n elements in \mathbb{A}^1 .

- (i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \dots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \rightarrow \mathbb{A}^1 \setminus S, (X_0, \dots, X_n) \mapsto X_0$$

is an isomorphism.

- (ii) Show that the ring of regular functions on $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to $k[t, \frac{1}{t}]$, the ring of polynomials in t and $\frac{1}{t}$.
- (iii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$, for instance by proving that their rings of regular functions are not isomorphic. (A different method will work just as well.)

Hint: Assume that

$$\Phi : A(X) \rightarrow k[t, t^{-1}]$$

is an isomorphism. Observe that $X_0 - a_i$ are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_0 - a_i) = t^{m_i}$ for some integers m_i . Furthermore, show that $m_i = \pm 1$, for instance by observing that $X_0 - a_i$ are not higher powers of any elements in $A(X)$. (This may be slightly trickier to prove.) Possibly using the fact that Φ is bijective, derive a contradiction.

Problem 2. (10 points.) Let $n \geq 2$. Consider the affine algebraic sets in \mathbb{A}^2 :

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

- (i) Show that

$$f : \mathbb{A}^1 \rightarrow Z_n, f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \rightarrow Z_n \setminus \{(0, 0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \rightarrow W_n, g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of \mathbb{A}^1 and W_n where g becomes an isomorphism.

- (ii) Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism between $Z_n \rightarrow W_n$.
- (iii) Assume that there exists an isomorphism

$$h : Z_n \rightarrow W_n$$

such that $h((0, 0)) = (0, 0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0, 0)\} \rightarrow W_n \setminus \{(0, 0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

- (iv) (*Entirely optional. Extra credit: 5 points*) Prove that h cannot exist even in the case when h may not send the origin to itself. (You may need to prove a stronger version of Problem 1.)
- (v) Show that Z_1 and W_1 are isomorphic. Write down an isomorphism between them.

Problem 3. (5 points.) Let $X \subset \mathbb{P}^2$ be a projective variety. A morphism $f : \mathbb{P}^1 \rightarrow X$ is a polynomial map

$$f([x : y]) = (f_0([x : y]), f_1([x : y]), f_2([x : y])),$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree, such that $f(\mathbb{P}^1) \subset X$.

Prove the following facts about lines and conics in projective plane:

- (i) For any line $L \subset \mathbb{P}^2$, there is a bijective morphism

$$f : \mathbb{P}^1 \rightarrow L.$$

- (ii) For any irreducible conic $C \subset \mathbb{P}^2$, there is a morphism

$$f : \mathbb{P}^1 \rightarrow C.$$

You may wish to change coordinates so that your conic has a convenient expression. Can you assume that f is bijective?

Remark: A curve X which is the image of a morphism $f : \mathbb{P}^1 \rightarrow X \subset \mathbb{P}^2$ is called a rational curve. This is the same thing as saying that X admits a polynomial parametrization. In particular, we showed that lines and conics in \mathbb{P}^2 are rational curves in \mathbb{P}^2 .

Problem 4. Part A. (2 points.) Show that any two distinct points in \mathbb{P}^2 lie on a unique line. Therefore, two distinct lines in \mathbb{P}^2 intersect in exactly one point.

Part B. (3 points.) Show that a line and an irreducible conic in \mathbb{P}^2 cannot intersect in 3 points. (We will see later that they intersect in exactly 2 points, if counted with multiplicity.)

Part C. (10 points.)

- (i) Four points in \mathbb{P}^2 are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if p_1, \dots, p_4 are points in general position, there exists a linear change of coordinates

$$T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$$

with

$$T([1 : 0 : 0]) = p_1, \quad T([0 : 1 : 0]) = p_2, \quad T([0 : 0 : 1]) = p_3, \quad T([1 : 1 : 1]) = p_4.$$

- (ii) Given five distinct points in \mathbb{P}^2 , no three of which are collinear, show that there is a unique irreducible projective conic passing through all five points. You may want to use part (i) to assume that four of the points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$.
- (iii) Deduce that two distinct irreducible conics in \mathbb{P}^2 cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

Remark: For any degree d , fix $3d - 1$ points in \mathbb{P}^2 in “general position”. You may ask how many rational curves of degree d in \mathbb{P}^2 pass through these $3d - 1$ points. We have shown above that there is $N_1 = 1$ line through 2 points, and $N_2 = 1$ conic through 5 points. The next few numbers are

$$N_3 = 12, N_4 = 620, N_5 = 87,304, N_6 = 26,312,976, N_7 = 14,616,808, 192.$$

Thus, there are 12 rational cubics through 8 points, 620 rational quartics through 11 points and so on. A general answer for arbitrary d was found in 1994 using ideas from physics/string theory. The area of algebraic geometry that computes these numbers is called enumerative geometry/Gromov-Witten theory.

Problem 5. (10 points.) Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to \mathbb{A}^1 . In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right)$$

where the pairs of polynomials (f, g) and (p, q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. By analyzing the factorization into irreducibles (in our case linear factors) of both numerators and denominators, conclude that $f, g, f-g, g-\lambda g$ must be perfect squares.

(ii) Prove the following:

Lemma: If f, g are polynomials in $k[t]$ such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f-g, f-\lambda g$ are perfect squares, then f and g must be constant.

Hint: Pick f and g such that $\max(\deg f, \deg g)$ is minimal among all pairs (f, g) which satisfy the requirement that $f, g, f-g, f-\lambda g$ are squares for some $\lambda \neq 0, 1$. Write $f = u^2, g = v^2$. Considering the factorizations of $f-g$ and $f-\lambda g$, prove that $u-v, u+v, u-\sqrt{\lambda}v, u+\sqrt{\lambda}v$ are also squares. Construct two constant coefficient linear combinations \tilde{u}, \tilde{v} of the polynomials u and v such that $\tilde{u}, \tilde{v}, \tilde{u}-\tilde{v}, \tilde{u}-\mu\tilde{v}$ are squares for some constant $\mu \in k \setminus \{0, 1\}$. Note that $\max(\deg \tilde{u}, \deg \tilde{v}) < \max(\deg f, \deg g)$ unless f, g are constant. Why is this a contradiction?

Remark: We will see later that any cubic curve can be written in the form

$$y^2 - x(x-1)(x-\lambda) = 0, \text{ or } y^2 - x^3 = 0 \text{ or } y^2 - x^2(x-1) = 0,$$

after a change of coordinates. The latter curves are Z_2 and W_2 in Problem 2, so they are birational to \mathbb{A}^1 .

(iii) Consider the elliptic curve $\overline{E}_\lambda \subset \mathbb{P}^2$:

$$y^2 z = x(x-z)(x-\lambda z).$$

Show that there are no nonconstant morphisms

$$\mathbb{P}^1 \rightarrow \overline{E}_\lambda \subset \mathbb{P}^2.$$

You may wish to prove first that any such morphism gives rise to a rational map

$$\mathbb{A}^1 \dashrightarrow E_\lambda.$$

Therefore, elliptic curves are not rational curves, and are not isomorphic to lines or conics.