Math 203, Problem Set 2. Due Monday October 15.

Hand in (at least) 3 problems from the list below.

For this problem set, you may assume that the ground field is algebraically closed.

1. Find the irreducible components of the affine algebraic set \(xz - y^2 = z^3 - x^5 = 0\) in \(\mathbb{A}^3\). What is the dimension of this affine algebraic set?

2. Which of the following algebraic sets are isomorphic and which ones are not:
   (i) \(\mathbb{A}^1\)
   (ii) \(Z(xy) \subset \mathbb{A}^2\)
   (iii) \(Z(x^2 + y^2) \subset \mathbb{A}^2\)
   (iv) \(Z(x^2 - y^3) \subset \mathbb{A}^2\)
   (v) \(Z(y - x^2, z - x^3) \subset \mathbb{A}^2\).

3. (Conics.) An algebraic set \(Z \subset \mathbb{A}^2\) defined by an irreducible polynomial \(f\) of degree 2 is called an irreducible conic.
   (i) Show that after an affine change of coordinates in \(\mathbb{A}^2\), any irreducible conic can be written in the form

   \[Y - X^2 = 0\text{ or } XY - 1 = 0.\]

   (ii) Let \(Z_1\) and \(Z_2\) be two distinct irreducible conics in \(\mathbb{A}^2\). Using (i), show that \(Z_1\) and \(Z_2\) intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

4. (Hartogs theorem and quasi-affine algebraic sets.) Show that the quasi-affine set \(X = \mathbb{A}^2 \setminus \{(0,0)\}\) is not isomorphic to an affine algebraic set:
   (i) Prove first the following algebro-geometric analogue of Hartog’s theorem in complex analysis: if \(f : X \to k\) is a regular function on \(X\), then \(f\) must be a polynomial in \(k[x,y]\), so it extends to \(\mathbb{A}^2\).
   (ii) If \(\Phi : Y \to X\) is an isomorphism between an affine algebraic set \(Y\) and \(X\), consider the composition \(\Psi = \iota \circ \Phi : Y \to \mathbb{A}^2\), where \(\iota : X \to \mathbb{A}^2\) is the inclusion. Show that the morphism \(\Psi\) induces an isomorphism on coordinate rings. Conclude that \(\Psi\) must be an isomorphism, hence \(\iota\) must be an isomorphism, a contradiction.

5. Let \(n \geq 2\), and let \(S = \{a_1, \ldots, a_n\}\) be a finite set with \(n\) elements in \(\mathbb{A}^1\).
   (i) Show that the quasi-affine set \(\mathbb{A}^1 \setminus S\) is isomorphic to an affine set. For instance, you may take \(X\) to be the affine algebraic set given by the equations

   \[X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.\]
Show that the projection onto the first coordinate
\[ \pi : X \to \mathbb{A}^1 \setminus S, \quad (X_0, \ldots, X_n) \mapsto X_0 \]
is an isomorphism.

(ii) Show that \( \mathbb{A}^1 \setminus S \) is not isomorphic to \( \mathbb{A}^1 \setminus \{0\} \) by proving that their rings of regular functions are not isomorphic.

**Hint:** Assume that
\[ \Phi : A(X) \to k[t, t^{-1}] \]
is an isomorphism. Observe that \( X_i \) are invertible elements in \( A(X) \) for all \( 1 \leq i \leq n \). Show that their images must be invertible in \( k[t, t^{-1}] \). Prove that this implies that \( \Phi(X_i) = t^{m_i} \) for some integers \( m_i \). Derive a contradiction by comparing \( \Phi(X_0 - a_i) \) for different values of \( i \).

6. (You may assume the result of problem 5.) Let \( n \geq 2 \). Consider the affine algebraic sets in \( \mathbb{A}^{2} \):
\[ Z_n = Z(y^n - x^{n+1}) \]
and
\[ W_n = Z(y^n - x^n(x + 1)). \]
Show that \( Z_n \) and \( W_n \) are birational but not isomorphic.

(i) Show that
\[ f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1}) \]
is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets
\[ \mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}. \]
Similarly, show that
\[ g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t). \]
is a morphism of affine algebraic sets. Find open subsets of \( \mathbb{A}^1 \) and \( W_n \) where \( g \) becomes an isomorphism.

(ii) Using (i), explain why \( Z_n \) and \( W_n \) are birational. Write down a birational isomorphism \( Z_n \leftrightarrow W_n \).

(iii) Assume that there exists an isomorphism
\[ h : Z_n \to W_n \]
such that \( h((0,0)) = (0,0) \). Observe that this induces an isomorphism between the open sets
\[ Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}. \]
Use part (i) and the previous problem to conclude this cannot be true if \( n \geq 2 \).

(iv) (Optional.) Repeat the argument above without the assumption that \( h \) sends the origin to itself. You may need to prove a stronger version of Problem 5.