Math 203, Problem Set 2. Due Friday, October 9.

For this problem set, you may assume that the ground field is algebraically closed.

1. Which of the following algebraic sets are isomorphic and which ones are not:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2+y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2-y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y-x^2, z-x^3) \subset \mathbb{A}^2$.

2. (*Conics.*) An algebraic set $\mathcal{Z} \subset \mathbb{A}^2$ defined by an irreducible polynomial f of degree 2 is called an irreducible conic.

(i) Show that after an affine change of coordinates in $\mathbb{A}^2$, any irreducible conic can be written in the form

$$Y - X^2 = 0 \text{ or } XY - 1 = 0.$$

(ii) Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

3. Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not isomorphic to an affine algebraic set.

Hint: If $\Phi : Y \to X$ is an isomorphism between an affine algebraic set Y and X, consider the composition $\Psi = \iota \circ \Phi : Y \to \mathbb{A}^2$, where $\iota : X \to \mathbb{A}^2$ is the inclusion. Explain that the morphism Ψ induces an isomorphism on coordinate rings. Conclude that Ψ must be an isomorphism, hence ι must be an isomorphism, a contradiction.

4. Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with n elements in $\mathbb{A}^1$.

(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \to \mathbb{A}^1 \setminus S,\ (X_0, \ldots, X_n) \mapsto X_0$$

is an isomorphism.

(ii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

Hint: Assume that

$$\Phi : A(X) \to k\left[t, t^{-1}\right]$$

is an isomorphism. Observe that X_i are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers m_i. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of i.

5. (*You may assume the result of Problem 4.*) Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and W_n where g becomes an isomorphism.

(ii) Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism $Z_n \dashrightarrow W_n$.

(iii) Assume that there exists an isomorphism

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) (*Optional.*) Repeat the argument above without the assumption that h sends the origin to itself. You may need to prove a stronger version of Problem 4.

6. (*Cubic curves are not rational.*) As a consequence of Problem 2, irreducible conics in $\mathbb{A}^2$ are rational. In this problem, we show that most cubic curves are not.

Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)}\right)$$

where the pairs of polynomials (f, g) and (p, q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. Conclude that $f, g, f-g, g-\lambda g$ must be perfect squares.

(ii) Conclude by proving the following:

Lemma: If f, g are polynomials in $k[t]$ without common factors and such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f-g, f-\lambda g$ are perfect squares, then f and g must be constant.

Hint: Descent. Write $f = u^2, g = v^2$. Considering $f - g$ and $f - \lambda g$, prove that $u-v, u+v, u-\mu v, u+\mu v$ are also squares for some constant $\mu \neq 0, 1$. Show that suitable $\tilde{u}, \tilde{v}$ obtained as a linear combination of u and v verify the lemma, yet they have smaller degree than $\max(\deg f, \deg g)$.

Remark: We will see later that any cubic curve can be written in the form

$$y^2 - x(x-1)(x-\lambda) = 0, \text{ or } y^2 - x^3 = 0 \text{ or } y^2 - x^2(x-1) = 0, .$$

The latter curves are Z_2 and W_2 in the previous problem, so they are birational to $\mathbb{A}^1$.