Math 203, Problem Set 4. Due Wednesday, November 4.

For this problem set, you may assume that the ground field is algebraically closed.

1. (*Quadrics are rational.*) Let $Q \subset \mathbb{P}^n$ be an irreducible complex projective quadric. We can record the coefficients of Q in a symmetric square matrix A such that

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \text{ for } \mathbf{x} \in \mathbb{P}^n.$$

We say that the quadric Q is nondegenerate if the matrix A is non-degenerate.

(i) Show that a non-degenerate irreducible quadric Q in $\mathbb{P}^3$ can be written as

$$xy = zw$$

after a suitable change of homogeneous coordinates. Conclude that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, hence Q is rational. You may wish to diagonalize A first.

(ii) In general, possibly using the projection from a point to a hyperplane, show that any non-degenerate irreducible quadric $Q \subset \mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$.

2. (*Joins.*) Let $\mathbb{G}(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in the previous homework. Show that:

(i) The set $\{(L, P) : P \in L\} \subset \mathbb{G}(1, n) \times \mathbb{P}^n$ is closed.

(ii) If $Z \subset \mathbb{G}(1, n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.

(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting X and Y is a closed subset of $\mathbb{P}^n$. It is called the join $J(X, Y)$ of X and Y.

3. (*Criterion for irreducibility.*) Assume that $f : X \to Y$ is a surjective morphism of projective algebraic sets such that Y is irreducible and all fibers of f are irreducible of the same dimension. Show that X is irreducible as well.

4. (*Intersections in projective space.*) Let X and Y be two subvarieties of $\mathbb{P}^n$. Show that if $\dim X + \dim Y \geq n$, then $X \cap Y$ is not empty.

Hint: Let H_1, H_2 be two disjoint linear subspaces of dimension n in $\mathbb{P}^{2n+1}$, and consider $X \subset H_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and $Y \subset H_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ as subvarieties of $\mathbb{P}^{2n+1}$. Show that the join $J(X, Y)$ in $\mathbb{P}^{2n+1}$ has dimension $\dim X + \dim Y + 1$. Then construct $X \cap Y$ as a suitable intersection of $J(X, Y)$ with $n + 1$ hyperplanes.

5. (*Lines on hypersurfaces.*) Let $d > 2n-3$. Show that a general degree d hypersurface in $\mathbb{P}^n$ contains no lines. (That is, show that at least one such hypersurface contains no lines. This implies that in fact a random such hypersurface contains no lines.)

Hint: View each hypersurface X as point in $\mathbb{P}^N$ for $N = \binom{n+d}{d} - 1$ by recording the coefficients of the defining equation. Consider the incidence correspondence $J = \{(L, X) : L \subset X\} \hookrightarrow \mathbb{G}(1, n) \times \mathbb{P}^N$. Calculate the the dimension of fibers under the projections to the two factors. Conclude that there is a nonempty open set in $\mathbb{P}^N$ corresponding to hypersurfaces without lines.