Math 203, Problem Set 5. Due Friday, November 13.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. (*Hyperelliptic curves.*) Let $a_1, \dots, a_{2g+1}$ be pairwise distinct constants. Find the singularities of the projective *hyperelliptic curve of genus g:*

$$y^2 z^{2g-1} = (x - a_1 z) \dots (x - a_{2g+1} z).$$

Remark: When $g = 1$, we get the elliptic curve $\overline{E}_\lambda$ already encountered before, by setting $a_1 = 0, a_2 = 1, a_3 = \lambda$.

2. (*Pencils of conics and singularities.*) Let Q_1 and Q_2 be two distinct *nonsingular* conics in $\mathbb{P}^2$. The family of conics

$$Q_{\lambda,\mu} = \lambda Q_1 + \mu Q_2$$

where $[\lambda : \mu] \in \mathbb{P}^1$ is called a *pencil* of conics.

(i) Recall that any conic $Q \subset \mathbb{P}^2$ determines and is determined by the symmetric matrix A of coefficients with

$$Q(\mathbf{x}) = \mathbf{x}^T \cdot A \cdot \mathbf{x}$$

Possibly by diagonalizing A (and therefore Q), show that

$$Q \text{ is singular if and only if } \det A = 0.$$

(ii) Letting $A_{\lambda,\mu}$ be the matrix associated to the conic $Q_{\lambda,\mu}$, show that $\det A_{\lambda,\mu}$ is a cubic polynomial in λ, μ. Prove that any pencil of conics contains (at most) 3 singular conics.

(iii) Let p_1, p_2, p_3, p_4 be points in $\mathbb{P}^2$ such that no three of them lie on a line. Show that the set of conics through p_1, p_2, p_3, p_4 is a pencil. (Feel free to change coordinates to prove this fact). What are the singular conics in this pencil? Can you draw them?

3. (*Singularities of hypersurfaces.*) Show that a *general* hypersurface of degree d in $\mathbb{P}^n$ is non-singular:

(i) For any hypersurface $\mathcal{Z}(f) \subset \mathbb{P}^n$ of degree d, view the coefficients of f as a point p_f in a large dimensional projective space $\mathbb{P}^N$ (This projective space is called *the moduli space* of degree d hypersurfaces). Let

$$X = \{(f, p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f\}.$$

Show that X is a projective algebraic set in $\mathbb{P}^N \times \mathbb{P}^n$.

(ii) Conclude that the image $\pi(X)$ of X under the projection onto $\mathbb{P}^N$ is a projective algebraic set. What is $\pi(X)$? Conclude that the subset of $\mathbb{P}^N$ corresponding to smooth hypersurfaces is open and *nonempty.*

Remark: This will prove that the hypersurface is singular provided that the coefficients of f satisfy certain polynomial relations. Therefore, if you pick f

randomly, these polynomial relations will most likely not be satisfied and your hypersurface is non-singular. This is the explanation of the word *general.*

4. (*Singularities of cubics.*)

(i) Show that any singular irreducible cubic in $\mathbb{P}^2$ is isomorphic to either the nodal or the cuspidal cubics:

$$y^2z = x^2(x+z) \text{ or } y^2z = x^3.$$

Hint: Assume the singularity is at $[0:0:1]$. Show that the cubic can be written as

$$(\text{quadratic polynomial in } x, y) \cdot z = Q(x,y),$$

where Q is a cubic polynomial in x, y. Change coordinates suitably and write the cubic as

$$y^2z = \widetilde{Q}(x,y) \text{ or } xyz = \widetilde{Q}(x,y).$$

Use the coordinate change $z \mapsto \lambda x + \mu y + \nu z$ to put the cubic into one of the forms

$$y^2z = (x+by)^3 \text{ or } xyz = (x+y)^3.$$

Conclude by performing one more change of coordinates.

(ii) Using (i), show that *irreducible cubics* in $\mathbb{P}^2$ can have at most 1 singular point. Exhibit a cubic in $\mathbb{P}^2$ with 3 singular points.

Remark: We will show later that an *irreducible* degree d curve in $\mathbb{P}^2$ has at most $\binom{d-1}{2}$ singular points.

5. (*Dual conics.*) Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x,y,z]$. Consider the morphism

$$\Phi : C \to \mathbb{P}^2, p \mapsto \left[\frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p)\right].$$

The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to C.

(i) Why is Φ a well-defined morphism? Find a geometric description of Φ, independent of coordinates.

(ii) If C is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic C is $ax^2 + by^2 + cz^2 = 0$. What is $\Phi(C)$?

(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

6. (*Analytic singularities.*) Consider the singular plane curves Z and W given by the equations

$$y^2 - x^2(x+1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain briefly why Z and W are not isomorphic. Explain that $(0,0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0,0)$ for Z and W? Sketch (the real points of) Z and W. Do Z and W look *alike* near the origin?

(ii) Show that there are *formal power series*

$$\tilde{x} = f_1 + f_2 + f_3 + \dots \text{ and}$$

$$\tilde{y} = g_1 + g_2 + g_3 \dots$$

in the variables x and y such that the equation of Z becomes

$$\tilde{x}\tilde{y} = 0.$$

Hint: Construct the degree i homogeneous parts f_i and g_i inductively. Show you can pick

$$f_1 = y - x, g_1 = x + y.$$

Next, you would need

$$f_2(x+y) + g_2(y-x) = -x^3.$$

Why can you construct f_2 and g_2? Continue in this fashion.

Remark: If we work over an arbitrary field k it doesnt make sense to ask if the power series $\tilde{x}$ and $\tilde{y}$ converge, hence the terminology *formal power series.* Convergence may be arranged if you work over the complex numbers, but you don't have to prove it.

Remark: It turns out the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is invertible *e.g.* you can solve for x, y in terms of formal power series in $\tilde{x}, \tilde{y}$. In fact, this statement is generally true about any power series

$$\tilde{x} = ax + by + \dots, \tilde{y} = cx + dy + \dots$$

provided that $ad - bc \neq 0$. Therefore, the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is a *formal* change of coordinates, establishing a *formal isomorphism* between Z and W. We say that Z and W are *analytically equivalent.*

Remark: Over the complex numbers, convergence may be arranged near the origin, if x, y are small, and thus the word *formal* may be replaced by *local analytic isomorphism* near the origin.

(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

Remark: It can be shown that any double point is analytically equivalent to the singularity $\tilde{y}^2 = \tilde{x}^r$, for some r. The case $r = 2$ corresponds to the case which concerned us above.