**Math 203, Problem Set 6. Due Monday, November 23.**

*For this problem set, you may assume that the ground field is $k = \mathbb{C}$.*

**1.** (*Normal varieties.*) Show that the quadric $x^2 + y^2 + z^2 = 0$ in $\mathbb{A}^3$ is normal.

*Hint:* Consider $\alpha \in K(X)$. Show that $\alpha = u + zv$ for $u, v \in k(x, y)$. Show that if $\alpha$ is integral then $u, v$ can be taken to be polynomials. To this end, show that the minimal polynomial of $\alpha$ over $k(x, y)$ is $T^2 - 2uT + (u^2 + v^2(x^2 + y^2)) = 0$ and use that its coefficients must be in $k[x, y]$ (why?)

**2.** (*Resolving curve singularities.*) Resolve the following $A_k$ plane curve singularity by subsequent blow-ups

$$y^2 - x^{k+1} = 0.$$

*Remark:* We have the following terminology on isolated "simple" singularities of hypersurfaces in $\mathbb{A}^{n+2}$:

- type $A_k$: $x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $D_k$: $x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_6$: $x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_7$: $x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_8$: $x^5 + y^3 + z_1^2 + \ldots + z_n^2 = 0$.

(The names suggest a connection with the Weyl groups of type $A, D, E$.)

**3.** (*Tangent cones.*) Let $X \subset \mathbb{A}^n$ be an affine variety and let $p \in X$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p}$. Show that the coordinate ring $A(C_{X,p})$ of the tangent cone of $X$ at $p$ is isomorphic to the graded algebra $\bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$.

*Hint:* Let $\mathfrak{i} \subset k[x_1, \ldots, x_n]$ be the ideal of $X$. Show that

$$k[x_1, \ldots, x_n]/\mathfrak{i}^{in} \to \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$$

given by $f \mapsto f^{(k)}|_X$ is an isomorphism.

**4.** (*Exceptional hypersurface.*) Consider the blowup of the affine variety $X \subset \mathbb{A}^n$ at $p \in X$. Show that the exceptional hypersurface is the projectivization of the tangent cone

$$E \cong \mathbb{P}(C_{X,p}).$$

You may want to generalize the argument we had in class for plane curves.

**5.** (*Cremona transformations.*) Consider the Cremona birational automorphism of $\mathbb{P}^2$ given by

$$C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1].$$

Let $\widetilde{\mathbb{P}^2}$ be the blowup of $\mathbb{P}^2$ at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$ where $C$ is undefined. Show that

(i) Show that $C$ extends to an isomorphism

$$\widetilde{C} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}.$$

(ii) Let $E_1, E_2, E_3$ be the exceptional lines for the blowup, and let $L_{ij}$ be the strict transform of the line through $P_i$ and $P_j$. Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under $\widetilde{C}$?

*Hint:* Show that the equations of the blowup $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are given by

$$x_0 y_0 = x_1 y_1 = x_2 y_2.$$

It may help to find the equations of the exceptional lines $E_1, E_2, E_3$ and those of the strict transforms $L_{23}, L_{12}, L_{13}$.

(iii) The group of automorphisms of the field of fractions in two variables

$$K(\mathbb{P}^2) \cong K(\mathbb{A}^2) \cong k(t_1, t_2)$$

is called the Cremona group. Therefore, the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$.

Explain that the Cremona transformation $C$ corresponds to the involution of $k(t_1, t_2)$ sending

$$(t_1, t_2) \to (t_1^{-1}, t_2^{-1}).$$

Furthermore, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group, in such a fashion that $-I_2$ corresponds to the Cremona transformation $C$.

*Remark:* The Cremona group is not yet fully understood (especially when the number of indeterminantes is bigger than 2).

**6.** (*Del Pezzo surfaces.*) From Andreas Gathmann's notes, read the proof that the blowup of $\mathbb{P}^2$ at two points is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. Do not hand in.