Math 203, Problem Set 7. Due Friday, December 4.

1. (*Degree of the Segre embedding.*) Show that the Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

has degree $\binom{n+m}{n}$.

2. (*Arithmetic genus.*) Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert polynomial χ_X. Define the arithmetic genus of X to be

$$p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1).$$

(i) Show that the genus of $\mathbb{P}^n$ is zero.

(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $p_a(X) = \binom{d-1}{n}$. In particular, a cubic in $\mathbb{P}^2$ has genus 1.

(iii) If X is a complete intersection of two surfaces of degree a and b in $\mathbb{P}^3$ then

$$p_a(X) = \frac{1}{2}ab(a+b-4) + 1.$$

In particular, intersection of two quadrics in $\mathbb{P}^3$ has genus 1.

Remark: To compare (ii) and (iii), recall that a cubic in $\mathbb{P}^2$ is isomorphic to an intersection of two quadrics in $\mathbb{P}^3$ as shown in a previous homework.

3. (*Enumerative geometry of lines.*) Given four general lines in $\mathbb{P}^3$, show that there are exactly 2 lines which intersect all four of them.

Hint: Recall that the Grassmannian $G(1,3)$ is a quadric in $\mathbb{P}^5$ via the Plücker embedding.

Remark: The number of lines in $\mathbb{P}^n$ which intersect $2(n-1)$ fixed general codimension 2 linear hyperplanes equals the Catalan number

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

4. (*Varieties of minimal degree.*) Let X be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree d and codimension c in $\mathbb{P}^n$.

(i) (Intersecting X with hyperplanes to cut down the dimension), show inductively that

$$d \geq c + 1.$$

(ii) Show that equality holds for rational normal curves in $\mathbb{P}^n$, and for the image $v(\mathbb{P}^2)$ of the Veronese embedding

$$v : \mathbb{P}^2 \to \mathbb{P}^5.$$

(iii) Can you classify the varieties of degree 2?

Remark: The Del Pezzo-Bertini theorem classifies all varieties for which equality holds in (i).