Problem 1. Which of the following algebraic sets are isomorphic:

(i) $\mathbb{A}^1$
(ii) $\mathbb{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathbb{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathbb{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathbb{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2$.

Answer: We claim that (i) and (v) are isomorphic, (ii) and (iii) are isomorphic, and (iv) is not isomorphic to any other algebraic sets.

- We check (i) and (v) are isomorphic. Note first that
  \[ \mathbb{Z}(y - x^2, z - x^3) = \{(t, t^2, t^3) \mid t \in \mathbb{C}\}. \]
  Define the morphisms
  \[ f : \mathbb{A}^1 \to \mathbb{Z}(y - x^2, z - x^3), f(t) = (t, t^2, t^3), \]
  and
  \[ g : \mathbb{Z}(y - x^2, z - x^3) \to \mathbb{A}^1, g(x, y, z) = x. \]
  Then
  \[ f \circ g = \text{identity, } g \circ f = \text{identity}. \]
  Thus $f$ and $g$ are inverse isomorphisms.

- We show (ii) and (iii) are isomorphic. Consider the morphisms
  \[ f : \mathbb{A}^2 \to \mathbb{A}^2, f(x, y) = (x + iy, x - iy), \]
  \[ g : \mathbb{A}^2 \to \mathbb{A}^2, g(x, y) = \left( \frac{x + y}{2}, \frac{x - y}{2i} \right). \]
  It is easy to check that
  \[ f \circ g = \text{identity, } g \circ f = \text{identity}. \]
  Moreover, letting
  \[ Z = \mathbb{Z}(x^2 + y^2) \text{ and } W = \mathbb{Z}(xy), \]
  we claim that
  \[ f(Z) \subset W, \ g(W) \subset Z. \]
  Indeed, if $(x, y) \in Z$, then
  \[ f(x, y) = (x - iy, x + iy) \in W \text{ since } (x - iy)(x + iy) = x^2 + y^2 = 0. \]
  Similarly, if $(x, y) \in W$, then
  \[ g(x, y) = \left( \frac{x + y}{2}, \frac{x - y}{2i} \right) \in Z, \text{ since } \left( \frac{x + y}{2} \right)^2 + \left( \frac{x - y}{2i} \right)^2 = 0. \]
Therefore, $f$ and $g$ establish isomorphisms between $Z$ and $W$. (Here we assume $k$ is not of characteristic 2. If so, this set is isomorphic to $\mathbb{A}^1$.)

- We show that (i) and (ii) are not isomorphic. Indeed, $Z(xy)$ is the union of two lines, hence it is a reducible affine set. On the other hand, $\mathbb{A}^1$ is irreducible, hence it cannot be isomorphic to $Z(xy)$.

- We show (ii) and (iv) cannot be isomorphic. This follows by the same argument observing that $Z(x^2 - y^5)$ is irreducible. Indeed, it suffices to prove that the polynomial $x^2 - y^5$ is irreducible. Assuming the contrary, write

$$x^2 - y^5 = f_1(x, y)f_2(x, y).$$

Regarding $f_1, f_2$ as polynomials in $x$ with coefficients in the integral domain $k[y]$, we conclude that $f_1, f_2$ can have degree at most 2 with respect to $x$. In fact, it is clear that the combination of degrees $(0, 2)$ cannot occur. If the degrees are 1, we may assume

$$f_1(x, y) = x - g(y), f_2(x, y) = x - h(y).$$

Then

$$x^2 - y^5 = x^2 - x(g(y) + h(y)) + g(y)h(y).$$

Therefore,

$$g(y) = -h(y), \text{ and } -y^5 = g(y)h(y) = -g(y)^2.$$

This is clearly impossible, proving our claim.

- We show that (i) and (iv) cannot be isomorphic. Letting $t$ be the coordinate of $\mathbb{A}^1$, we show that there cannot be an isomorphism

$$\Phi : k[x, y]/(x^2 - y^5) \to k[t].$$

Indeed, set

$$\Phi(x) = p, \Phi(y) = q.$$

We must have

$$p^2 = q^5.$$ 

This implies that all irreducible factors appearing in $q$ have even exponent, so

$$q = r^2, p = r^5$$

for some polynomial $r$. Note that $r$ cannot be constant since otherwise the image of $\Phi$ would have to consist in constant polynomials.

Now since $\Phi$ is surjective, there is a polynomial

$$f = \sum_{i,j} a_{ij}x^iy^j.$$
such that $\Phi(f) = r$. This means that

$$\sum_{ij} a_{ij} r^{5i+2j} = r.$$ 

In particular, since the left hand side must be divisible by $r$, we have $a_{00} = 0$. However, since $5i + 2j \geq 2$ for $(i, j) \neq (0, 0)$, the left hand side is in fact divisible by $r^2$, so it cannot equal $r$. This contradiction shows that an isomorphism $\Phi$ cannot exist.

\[ \square \]

**Problem 2.** An algebraic set $C \subset \mathbb{A}^2$ defined by an irreducible polynomial $F$ of degree 2 is called an irreducible conic.

(i) Show that any irreducible conic is isomorphic to

$$Y - X^2 = 0 \text{ or } XY - 1 = 0.$$ 

(ii) Let $C_1$ and $C_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that $C_1$ and $C_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

**Answer:** Let

$$F(x, y) = ax^2 + by^2 + cxy + dx + ey + f.$$ 

We will show that after a linear/affine change of coordinates the conic can be written as

$$XY - 1 = 0 \text{ or } Y - X^2 = 0.$$ 

We discuss the following cases.

**Case 1.** If $a = b = 0$, then

$$F(x, y) = cxy + dx + ey + f = c\left(xy + \frac{d}{c}x + \frac{e}{c}y\right) + f = c\left(x + \frac{e}{c}\right)\left(y + \frac{d}{c}\right) + \hat{f}$$

If $\hat{f} = 0$, $F$ is reducible, which is not allowed. Therefore $\hat{f} \neq 0$. Let

$$X = -\frac{e}{\hat{f}}\left(x + \frac{e}{c}\right) \quad \text{and} \quad Y = y + \frac{d}{c},$$

so that

$$F = -\hat{f}XY + \hat{f}.$$ 

Thus $F(x, y) = 0$ implies

$$XY - 1 = 0.$$ 

After an affine change of coordinates, the conic $Z(F)$ becomes

$$XY - 1 = 0.$$
Case 2. If either $a$ or $b$ is not 0, without loss of generality, we may assume $a \neq 0$. Then,

$$F = ax^2 + by^2 + cxy + dx + ey + f = a \left( x + \frac{c}{2a}y \right)^2 + by^2 + dx + ey + f$$

Let

$$x_1 = \sqrt{a} \left( x + \frac{c}{2a}y \right) \quad \text{and} \quad y_1 = y,$$

(choose any one of the square roots). There exist constants $\tilde{d}, \tilde{e}, \tilde{f}$ such that

$$F = x_1^2 + by_1^2 + \tilde{d}x_1 + \tilde{e}y_1 + \tilde{f} = \left( x_1 + \frac{\tilde{d}}{2} \right)^2 + by_1^2 + \tilde{e}y_1 + \tilde{f}.$$

Let

$$x_2 = x_1 + \frac{\tilde{d}}{2}.$$ 

Subcase (i): If $b = 0$,

$$F = x_2^2 + \tilde{e}y_1 + \tilde{f}.$$ 

We claim $\tilde{e} \neq 0$ because otherwise

$$F = (x_2 + i\sqrt{\tilde{f}})(x_2 - i\sqrt{\tilde{f}})$$

is reducible. Let

$$X = x_2, \quad Y = -(\tilde{e}y_1 + \tilde{f}).$$

Then

$$F = X^2 - Y.$$ 

Subcase (ii): If $b \neq 0$, let

$$y_2 = \sqrt{by_1} + \frac{\tilde{e}}{2\sqrt{b}}$$

so that

$$F = x_2^2 + y_2^2 + g.$$ 

Letting

$$X = \sqrt{-g}(x_2 + iy_2) \quad \text{and} \quad Y = \sqrt{-g}(x_2 - iy_2),$$

we have

$$F = -g(XY - 1).$$

Therefore, the conic $Z(F)$ can be written in the form

$$XY - 1 = 0$$

after an affine change of coordinates.
(ii) Suppose $C_1 = \mathcal{Z}(f)$ and $C_2 = \mathcal{Z}(g)$ for some polynomials $f, g \in \mathbb{C}[X, Y]$. Using (i), we can change coordinates to achieve 

$$f = XY - 1 \text{ or } f = Y - X^2.$$ 

Let us write 

$$g = aX^2 + bY^2 + cXY + dX + eY + f.$$ 

Let $(x, y) \in C_1 \cap C_2$, i.e. 

$$f(x, y) = g(x, y) = 0.$$ 

**Case 1.** If $f = XY - 1$, then 

$$y = \frac{1}{x} \text{ and } x \neq 0.$$ 

Substituting into $g$, we get 

$$0 = g(x, 1/x) = ax^2 + \frac{b}{x^2} + cx \cdot \frac{1}{x} + dx + e + f.$$ 

Multiplying by $x^2$, we obtain 

$$ax^4 + b + cx^3 + dx^2 + ex + fx^2 = 0.$$ 

This degree 4 polynomial has at most 4 roots. Therefore, $C_1 \cap C_2$ contains at most 4 points.

**Case 2.** If $f = Y - X^2$, then 

$$y = x^2.$$ 

Substituting into $g$, we get 

$$0 = g(x, x^2) = ax^2 + bx^4 + cx^3 + dx + ex^2 + f.$$ 

Thus $x$ satisfies a polynomial of degree at most 4, so there are at most 4 values for $x$. As $y = x^2$, it follows that $C_1 \cap C_2$ contains at most 4 points.

The following examples show that the number of intersection points can be either 0, 1, 2, 3, 4:

(i) The conics $XY - 1 = 0$ and $X^2 + XY - 1 = 0$ intersect in 0 points.

(ii) The conics $Y - X^2 = 0$ and $Y^2 + Y - X^2 = 0$ intersect in 1 point.

(iii) The conics $Y - X^2 = 0$ and $Y^2 + XY + Y - X^2 = 0$ intersect in 2 points.

(iv) The conics $XY - 1 = 0$ and $Y - X^2 = 0$ intersect in 3 points.

(v) The conics $X - Y^2 = 0$ and $Y - X^2 = 0$ intersect in 4 points.

\[\square\]

**Problem 3.** Show that $X = \mathbb{A}^2 \setminus \{(0,0)\}$ cannot be isomorphic to an affine algebraic set:

(i) Show that if $f : X \to k$ is a regular function on $X$, then $f$ must be a polynomial in $k[x, y]$. 


If  \( \Phi : Y \to X \) is an isomorphism between an affine algebraic set  \( Y \) and  \( X \), consider the composition  \( \Psi = \iota \circ \Phi \), where  \( \iota : X \to \mathbb{A}^2 \) is the inclusion. Show that the morphism  \( \Psi \) induces an isomorphism on coordinate rings. Conclude that  \( \Psi \) must be an isomorphism, hence  \( \iota \) must be an isomorphism, which must be a contradiction.

**Answer:**

(i) Suppose  \( f \) is a regular function on  \( X \). In particular,  \( f \) must be regular over the basic open set  \( \mathbb{A}^2 \setminus \{ x = 0 \} \) hence it must be of the form

\[
f(x, y) = \frac{p(x, y)}{x^n}
\]

for  \( n \geq 0 \) and some polynomial  \( p \). In a similar fashion,  \( f \) must be regular over  \( \mathbb{A}^2 \setminus \{ y = 0 \} \) hence it must be of the form

\[
f(x, y) = \frac{q(x, y)}{y^m}
\]

for  \( m \geq 0 \). Thus

\[
f(x, y) = \frac{p(x, y)}{x^n} = \frac{q(x, y)}{y^m}
\]

which gives  \( n = m = 0 \) and  \( f \) must be a polynomial.

(ii) Note that  \( g \) is a regular function on  \( Y \) \( \Leftrightarrow g \circ \Phi^{-1} \) is a regular function on  \( X \) \( \Leftrightarrow g \circ \Phi^{-1} \in k[x, y] \) by (i). Letting  \( P = g \circ \Phi^{-1} \), we have

\[
g = P \circ \Psi
\]

for some polynomial  \( P \in k[x, y] \). Thus,  \( \Psi^* \) induces an isomorphism between the coordinate ring of  \( Y \) and  \( k[x, y] \) proving that

\[
\Psi : Y \cong \mathbb{A}^2
\]

is an isomorphism. In turn, this means  \( \iota : X \cong \mathbb{A}^2 \) is an isomorphism. This is impossible because  \( \iota : X \to \mathbb{A}^2 \) is not bijective.

\[\square\]

**Problem 4.** Let  \( n \geq 2 \), and let  \( S = \{ a_1, \ldots, a_n \} \) be a finite set with  \( n \) elements in  \( \mathbb{A}^1 \).

(i) Show that the quasi-affine set  \( \mathbb{A}^1 \setminus S \) is isomorphic to an affine set. For instance, you may take  \( X \) to be the affine algebraic set given by the equations

\[
X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.
\]

Show that the projection onto the first coordinate

\[
\pi : X \to \mathbb{A}^1 \setminus S, \ (X_0, \ldots, X_n) \mapsto X_0
\]

is an isomorphism.

(ii) Show that  \( \mathbb{A}^1 \setminus S \) is not isomorphic to  \( \mathbb{A}^1 \setminus \{ 0 \} \) by proving that their rings of regular functions are not isomorphic.
Answer: (i) If \((X_0, \ldots, X_n) \in X\) we have \(X_i(X_0 - a_i) = 1\) hence \(X_0 \neq a_i\) for all \(1 \leq i \leq n\). Therefore
\[
\pi : X \to \mathbb{A}^1 \setminus S
\]
is well defined. Let
\[
\phi : \mathbb{A}^1 \setminus S \to X, t \mapsto \left( t, \frac{1}{t - a_1}, \ldots, \frac{1}{t - a_n} \right).
\]
Both \(\pi\) and \(\phi\) are rational maps, regular everywhere. It is clear that
\[
\pi \circ \phi = \text{identity} \quad \text{and} \quad \phi \circ \pi = \text{identity}.
\]
Therefore, \(\pi\) and \(\phi\) are inverse isomorphisms.

(ii) Assume there is an isomorphism
\[
\Phi : A(X) \to k[t, t^{-1}].
\]
Since
\[
X_i(X_0 - a_i) = 1 \quad \text{in} \ A(X),
\]
it follows that
\[
\Phi(X_i)\Phi(X_0 - a_i) = 1.
\]
Writing
\[
\Phi(X_i) = \frac{g_i(t)}{t^{\alpha_i}}, \quad \Phi(X_0 - a_i) = \frac{h_i(t)}{t^{\beta_i}},
\]
for some polynomials \(g_i, h_i\), we obtain
\[
g_i(t)h_i(t) = t^{\alpha_i + \beta_i}.
\]
This implies that \(h_i\) is of the form \(ct^m\), or equivalently
\[
\Phi(X_0 - a_i) = ci^m
\]
for some \(m_i \in \mathbb{Z}\) and \(c_i \in k\). Subtracting the relations for \(i\) and \(j\), it follows that
\[
a_j - a_i = \Phi(a_j - a_i) = c_j t^{m_j} - c_i t^{m_i}.
\]
Comparing degrees, we see that this implies \(m_i = m_j = 0\), as \(a_i \neq a_j\) for \(i \neq j\).
In turn, we obtain
\[
\Phi(X_0 - a_i) = c_i.
\]
Since \(\Phi(c_i) = c_i\), we contradicted the injectivity of \(\Phi\). This shows that \(\Phi\) cannot be an isomorphism completing the proof.

\[\square\]

Problem 5. Let \(n \geq 2\). Consider the affine algebraic sets in \(\mathbb{A}^2\):
\[
Z_n = \mathbb{Z}(y^n - x^{n+1})
\]
and
\[
W_n = \mathbb{Z}(y^n - x^n(x + 1)).
\]
Show that \(Z_n\) and \(W_n\) are birational but not isomorphic.
(i) Show that

\[ f : \mathbb{A}^1 \to \mathbb{Z}^n, \quad f(t) = (t^n, t^{n+1}) \]

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

\[ \mathbb{A}^1 \setminus \{0\} \to \mathbb{Z}^n \setminus \{(0,0)\}. \]

Similarly, show that

\[ g : \mathbb{A}^1 \to \mathbb{W}^n, \quad g(t) = (t^n - 1, t^{n+1} - t) \]

is a morphism of affine algebraic sets. Find open subsets of \( \mathbb{A}^1 \) and \( \mathbb{W}^n \) where \( g \) becomes an isomorphism.

(ii) Using (i), explain why \( \mathbb{Z}^n \) and \( \mathbb{W}^n \) are birational. Write down a birational isomorphism between \( \mathbb{Z}^n \to \mathbb{W}^n \).

(iii) Assume that there exists an isomorphism

\[ h : \mathbb{Z}^n \to \mathbb{W}^n \]

such that \( h((0,0)) = (0,0) \). Observe that this induces an isomorphism between the open sets

\[ \mathbb{Z}^n \setminus \{(0,0)\} \to \mathbb{W}^n \setminus \{(0,0)\}. \]

Use part (i) and the previous problem to conclude this cannot be true if \( n \geq 2 \).

Answer: 

(i) It is clear that

\[ f : \mathbb{A}^1 \to \mathbb{Z}^n, \quad t \to (t^n, t^{n+1}) \]

is a well defined morphism. Consider the morphism

\[ f^{-1} : \mathbb{Z}^n \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\} \]

given by

\[ (x, y) \to \frac{y}{x}. \]

A direct computation shows that \( f^{-1} \) is the inverse morphism of \( f \). Similarly,

\[ g : \mathbb{A}^1 \to \mathbb{W}^n, \quad t \to (t^n - 1, t^{n+1} - t) \]

is a well defined morphism. Its inverse morphism is

\[ g^{-1} : \mathbb{W}^n \setminus \{0\} \to \mathbb{A}^1 \setminus S, \quad (x, y) \to \frac{y}{x}. \]

Here, \( S \) is the set of all \( n \) roots of unity. The two morphisms \( g \) and \( g^{-1} \) establish an isomorphism between

\[ \mathbb{A}^1 \setminus S \to \mathbb{W}^n \setminus \{0\}. \]

(ii) Part (i) shows that both \( \mathbb{Z}^n \) and \( \mathbb{W}^n \) are birational to \( \mathbb{A}^1 \) so they are birational to each other. An explicit birational isomorphism is

\[ g \circ f^{-1} : \mathbb{Z}^n \longrightarrow \mathbb{W}^n. \]
A direct computation shows
\[ g \circ f^{-1}(x, y) = \left( \frac{y^n}{x^n}, \frac{y^{n+1}}{x^{n+1}} - \frac{y}{x} \right). \]

This expression could also be further simplified using the equation of \( Z_n \) as \((x - 1, y - \frac{y}{x})\).

(iii) If \( h : Z_n \to W_n \) is an isomorphism sending the origin to itself, then
\[ h : Z_n \setminus \{0\} \to W_n \setminus \{0\} \]
is also an isomorphism. By part (i), \( g^{-1} \circ h \circ f \) induces an isomorphism between the quasi-affine sets
\[ \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus S. \]
Such an isomorphism cannot exist by Problem 1.

\[ \square \]

**Problem 6.** Let \( \lambda \in k \setminus \{0, 1\} \). Consider the cubic curve \( E_\lambda \subset \mathbb{A}^2 \) given by the equation
\[ y^2 - x(x - 1)(x - \lambda) = 0. \]

Show that \( E_\lambda \) is not birational to \( \mathbb{A}^1 \). In fact, show that there are no non-constant rational maps
\[ F : \mathbb{A}^1 \to E_\lambda. \]

(i) Write
\[ F(t) = \left( \frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right) \]
where the pairs of polynomials \((f, g)\) and \((p, q)\) have no common factors. Conclude that
\[ \frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^2} \]
is an equality of fractions that cannot be further simplified. Conclude that \( f, g, f - g, g - \lambda g \) must be perfect squares.

(ii) Prove the following:

**Lemma:** If \( f, g \) are polynomials in \( k[t] \) without common factors and such that there is a constant \( \lambda \neq 0, 1 \) for which \( f, g, f - g, f - \lambda g \) are perfect squares, then \( f \) and \( g \) must be constant.

**Answer:** (i) We have
\[ \frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^2}. \]
Since \( p, q \) are relatively prime, the right hand side cannot be further simplified. Similarly, \( f, g, f - g, f - \lambda g \) cannot have any common factors. Indeed, a common factor for instance of \( f \) and \( f - g \), will necessarily have to divide \( f - (f - g) = g \).
as well. But this is impossible since \( f \) and \( g \) are coprime. Therefore the right hand side cannot be further simplified as well. Thus, for some constant \( a \in k \), we must have

\[
ap^2 = f(f - g)(f - \lambda g), \quad aq^2 = g^3.
\]

The exponents of the irreducible factors of the left hand sides of the above equations must be even. Therefore, the same must be true about the right hand side. This immediately implies that \( g \) is a square. But since \( f, f - g, f - \lambda g \) have no common factors, it follows that the exponents of the irreducible factors of each \( f, f - g \) and \( f - \lambda g \) must be even as well. Thus \( f, f - g, f - \lambda g \) must be squares.

(ii) Pick \( f \) and \( g \) such that \( \max(\deg f, \deg g) \) is minimal among all pairs \((f, g)\) which satisfy the requirement that \( f, g, f - g, f - \lambda g \) are squares for some \( \lambda \neq 0, 1 \). We may assume that \( f, g \) are coprime since otherwise we can reduce their degree by dividing by their gcd which is also a square. Write

\[
f = u^2, \quad g = v^2,
\]

where \( u, v \) are coprime. Then

\[
f - g = (u - v)(u + v)
\]

is a square. Note that \( u - v \) and \( u + v \) cannot have common factors since such factors will have to divide both

\[
\frac{1}{2}((u - v) + (u + v)) = u \quad \text{and} \quad \frac{1}{2}((u - v) - (u + v)) = v
\]

which is assumed to be false. Thus \( u - v \) and \( u + v \) are coprime, and since their product \( f - g \) is a square, it follows that \( u - v, u + v \) must be square. The same argument applied to

\[
f - \lambda g = (u - \sqrt{\lambda}v)(u + \sqrt{\lambda}v)
\]

shows that \( u - \sqrt{\lambda}v, u + \sqrt{\lambda}v \) are squares. Let

\[
\tilde{u} = \frac{1 + \sqrt{\lambda}}{2}(u + v), \quad \tilde{v} = \frac{\sqrt{\lambda} - 1}{2}(u - v).
\]

Clearly, \( \tilde{u}, \tilde{v} \) are squares. A direct computation shows that

\[
\tilde{u} - \tilde{v} = u + \sqrt{\lambda}v,
\]

which is also a square. Finally,

\[
\tilde{u} - \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right)^2 \tilde{v} = 1 + \sqrt{\lambda}(u - \sqrt{\lambda}v)
\]

is a square. Setting

\[
\mu = \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right)^2,
\]
we see that $\tilde{u}, \tilde{v}, \tilde{u} - \tilde{v}, \tilde{u} - \mu \tilde{v}$ are squares. Note that $\mu \neq 0, 1$ for $\lambda \neq 0, 1$.

Furthermore,

$$\max(\deg \tilde{u}, \deg \tilde{v}) = \frac{1}{2} \max(\deg f, \deg g).$$

Unless $f$ and $g$ are irreducible, this contradicts the assumption that $f, g$ are of minimal degree.

$\square$