Problem 1. Let $a_1, \ldots, a_{2g+1}$ be pairwise distinct constants. Find the singularities of the projective hyperelliptic curve of genus $g$:

$$y^2z^{2g-1} = (x - a_1z)\ldots(x - a_{2g+1}z).$$

**Answer:** Let

$$f(x,y,z) = y^2z^{2g-1} - (x - a_1z)\ldots(x - a_{2g+1}z).$$

Then $f$ is singular at $p$ if and only if

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Since

$$\frac{\partial f}{\partial y} = 2yz^{2g-1},$$

we see that if $p = [x : y : z]$ is a singular point then $y = 0$ or $z = 0$.

If $y = 0$, from $f(p) = 0$ we obtain $x = a_iz$ for some $i$. Because we are free to scale the coordinates in $\mathbb{P}^2$, $p = [a_i : 0 : 1]$. We compute

$$\frac{\partial f}{\partial x} = -\sum_{k=1}^{2g+1} \prod_{j \neq k}(x - a_jz).$$

Similarly,

$$\frac{\partial f}{\partial z} = (2g - 1)y^2z^{2g-2} + \sum_{k=1}^{2g+1} a_k \prod_{j \neq k}(x - a_jz).$$

Thus

$$\frac{\partial f}{\partial x}(p) = \prod_{j \neq i}(a_j - a_i) \neq 0$$

since $a_i$ are distinct. Thus $[a_i : 0 : 1]$ are not singular points.

If $z = 0$, $f(x,y,z) = x^{2g+1} = 0$, which is only possible when $x = 0$. Then $p = [0 : 1 : 0]$. The formulas above show

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0,$$

when $g \geq 2$ and $\frac{\partial f}{\partial z}(p) \neq 0$ when $g = 1$. Thus $[0 : 1 : 0]$ is the only singular point when $g \geq 2$. \hfill \square

Problem 2. Let $Q_1$ and $Q_2$ be two distinct nonsingular conics in $\mathbb{P}^2$. The family of conics

$$Q_{\lambda, \mu} = \lambda Q_1 + \mu Q_2$$

where $[\lambda : \mu] \in \mathbb{P}^1$ is called a pencil of conics.
(i) Recall that any conic \( Q \subset \mathbb{P}^2 \) determines and is determined by the symmetric matrix \( A \) of coefficients with
\[ Q([x : y : z]) = [x \ y \ z]^T A [x \ y \ z]. \]
Possibly by diagonalizing \( A \) (and therefore \( Q \)), show that
\( Q \) is singular if and only if \( \det A = 0 \).

(ii) Letting \( A_{\lambda,\mu} \) be the matrix associated to the conic \( Q_{\lambda,\mu} \), show that \( \det A_{\lambda,\mu} \) is a cubic polynomial in \( \lambda, \mu \). Prove that any pencil of conics contains (at most) 3 singular conics.

(iii) Let \( p_1, p_2, p_3, p_4 \) be points in \( \mathbb{P}^2 \) such that no three of them lie on a line. Show that the set of conics through \( p_1, p_2, p_3, p_4 \) is a pencil. (Feel free to change coordinates to prove this fact). What are the singular conics in this pencil?

**Answer:**

(i) Since \( A \) is symmetric, by the spectral theorem there exists an orthonormal basis of eigenvectors. Let \( C \) be the matrix whose columns are this basis, so that \( C^T AC \) is diagonal. We make the change of coordinates \( X^{\text{new}} = C^T X \).

Under these coordinates, we have
\[ Q = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2, \]
where the \( \lambda_i \) are the eigenvalues of \( A \). Then a point \([x : y : z]\) is singular if and only if the three partial derivatives of \( Q \) vanish, that is
\[ 2\lambda_1 x = 2\lambda_2 y = 2\lambda_3 z = 0. \]

If such a point exists, then at least one of \( x, y, z \) is nonzero; WLOG let \( x \neq 0 \). Then the last equality gives \( \lambda_1 = 0 \), hence
\[ \det A = \lambda_1\lambda_2\lambda_3 = 0. \]
Conversely, if \( \det A = 0 \), then one of the \( \lambda_i \) must be zero; WLOG let \( \lambda_1 = 0 \). Then \([1 : 0 : 0]\) is a singularity.

(ii) If \((a_{ij})\) and \((b_{ij})\) are respectively the matrices associated to \( Q_1 \), and \( Q_2 \), then
\[ A_{\lambda,\mu} = (\lambda a_{ij} + \mu b_{ij}). \]

In calculating the determinant, each term is the product of three such linear terms in \( \lambda \) and \( \mu \), hence the determinant is a cubic polynomial in \( \lambda \) and \( \mu \).

Note that \( Q_{\lambda,\mu} \) is singular if and only if
\[ \det A_{\lambda,\mu} = 0. \]
Furthermore, for \( \lambda \neq 0 \) we have \( \det A_{\lambda,0} = \lambda^3 \det Q_1 \neq 0 \) since \( Q_1 \) is nonsingular and by (i). That is, \( \det A_{\lambda,\mu} \) contains a \( \lambda^3 \) term, and similarly it contains a \( \mu^3 \) term. Hence, if \( \det A_{\lambda,\mu} \) equals zero and either \( \lambda \) or \( \mu \) is zero, then the other
must be as well, which yields no cubic. Thus, we may assume $\lambda, \mu$ are nonzero, and we may further assume $\mu = 1$ (since $Q_{\lambda,\mu} = Q_{c\lambda,c\mu}$). Since $\det A_{\lambda,1} = 0$ is a cubic equation, it has at most three solutions, and therefore any pencil of conics contains at most three singular conics.

(iii) Change coordinates such that

$$p_1 = [1 : 0 : 0] \quad p_2 = [0 : 1 : 0] \quad p_3 = [0 : 0 : 1] \quad p_4 = [1 : 1 : 1]$$

Let $Q$ be the equation for a conic passing through these points. $Q$ cannot contain an $x^2$ term, as then $Q([1 : 0 : 0])$ would not equal zero, and likewise $Q$ cannot contain an $y^2$ or $z^2$ term. Hence, $Q$ must be of the form

$$a xy + b yz + c xz,$$

which passes through the first three points for any $a, b, c$. The conic passes through $p_4$ as well if and only if $a + b + c = 0$.

The subspace

$$\{(a, b, c) : a + b + c = 0\}$$

has dimension two with basis

$$\{(1, 1, -2), (1, -2, 1)\}.$$ 

Therefore, all such conics $Q$ can be written as

$$\lambda(xy + yz - 2xz) + \mu(xy - 2yz + xz) = 0$$

for some $\lambda, \mu$.

To show that this gives a pencil, we must check that the conics

$$f = xy + yz - 2xz \quad \text{and} \quad g = xy - 2yz + xz$$

are nonsingular. Taking derivatives,

$$\frac{\partial f}{\partial x} = y - 2x \quad \frac{\partial f}{\partial y} = x + z \quad \frac{\partial f}{\partial z} = y - 2z$$

If all are to vanish, then from the $x$- and $z$-partials we conclude $x = z$, but then the $y$-partial does not vanish unless $x = y = z = 0$, which is impossible in $\mathbb{P}^2$. Therefore, $f$ gives a nonsingular conic, and similarly $g$ does as well.

From (ii), there are at most three nonsingular conics in the pencil. Three such conics are

$$xy - yz, \quad xy - xz, \quad yz - xz,$$

which have singularities for instance at

$$[1 : 0 : 1], \quad [0 : 1 : 1], \quad [1 : 1 : 0].$$

Therefore, they are the only ones.
Geometrically, these conics are the three union of lines joining the four points. For instance the singular conic \( xy - yz \) is the union of the lines \( p_1p_3 \) and \( p_2p_4 \). The singular conic \( xy - xz \) is the union of the lines \( p_1p_4 \) and \( p_2p_3 \). Finally the singular conic \( yz - xz \) is the union of the lines \( p_1p_3 \) and \( p_2p_4 \). 

\[ \square \]

**Problem 3.** Show that a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is non-singular:

(i) For any hypersurface \( Z(f) \subset \mathbb{P}^n \) of degree \( d \), view the coefficients of \( f \) as a point \( p_f \) in a large dimensional projective space \( \mathbb{P}^N \) (This projective space is called the moduli space of degree \( d \) hypersurfaces). Let

\[
X = \{ (f, p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f \}.
\]

Show that \( X \) is a projective algebraic set in \( \mathbb{P}^N \times \mathbb{P}^n \).

(ii) Conclude that the image \( \pi(X) \) of \( X \) under the projection onto \( \mathbb{P}^N \) is a projective algebraic set. What is \( \pi(X) \)? Conclude that the subset of \( \mathbb{P}^N \) corresponding to smooth hypersurfaces is open and nonempty.

**Answer:**  

(i) Let 

\[
f = \sum_I a_I X^I,
\]

where \( I \) is a multi-index. Then \( a_I \) will be the coordinates of the point \( p_f \) in \( \mathbb{P}^N \).

Let \( p = [x_0 : x_1 : \ldots : x_n] \), the condition \( f \) is singular at \( p \) is equivalent to

\[
f(p) = \frac{\partial f}{\partial X_i}(p) = 0 \quad \text{for all } i \quad \text{and}
\]

\[
\sum_I a_I x^I = \frac{\partial(\sum_I a_I X^I)}{\partial X_i}(p) = 0.
\]

Let \( a_I \) vary in \( \mathbb{P}^N \) and \( x \) vary in \( \mathbb{P}^n \), the equations above can be viewed as equations of \( a_I \) and \( x \) in \( \mathbb{P}^N \times \mathbb{P}^n \) which are bi-homogeneous in the variables. Therefore \( X \) is projective algebraic.

(ii) As shown in class, the projection \( \pi \) is a closed map e.g. \( \pi(X) \) is a projective algebraic set. Note that

\[
\pi(X) = \{ f \mid f \text{ is a nonsingular homogeneous degree } d \text{ polynomial} \}
\]

The complement of \( \pi(X) \) is open, therefore nonsingular homogeneous degree \( d \) polynomial form an open set in the moduli space.

To show non-emptyness, observe that the hypersurface 

\[
f = X_0^d + \ldots + X_n^d
\]

has no singularities. Indeed, all derivatives of \( f \) are \( dX_i^{d-1} \) which do not have a common vanishing in \( \mathbb{P}^n \).
Problem 4.  

(i) Show that any singular irreducible cubic in \( \mathbb{P}^2 \) is isomorphic to either the nodal or the cuspidal cubics:

\[ y^2z = x^2(x+z) \text{ or } y^2z = x^3. \]

(ii) Using (i), show that irreducible cubics in \( \mathbb{P}^2 \) can have at most 1 singular point. Exhibit a cubic in \( \mathbb{P}^2 \) with 3 singular points.

Answer:  

(i) Assume the singularity is at \([0:0:1]\) and let \( f \) be the polynomial giving the cubic. Then since

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0, \]

\( f \) may not contain an \( xz^2 \), \( yz^2 \) or \( z^3 \) terms. That is, \( z \) does not appear to a power higher than 1 in \( f \), and so we may write the cubic as

\[ (\text{quadratic polynomial in } x, y) \cdot z = Q(x, y) \]

where \( Q \) is a cubic polynomial in \( x, y \).

The quadratic polynomial is either the square of a linear term or the product of two distinct linear terms in \( x \) and \( y \). In the first case, let \( y^{\text{new}} \) be that linear term, or in the second let \( x^{\text{new}} \) and \( y^{\text{new}} \) be the two linear terms. We obtain

\[ y^2z = Q(x, y) \quad \text{or} \quad xyz = Q(x, y) \]

Consider the first case, \( y^2z = Q(x, y) \). We note first that \( Q \) must contain an \( x^3 \) term, as otherwise both sides are divisible by \( y \) contradicting that the conic is irreducible. First make a scaling change of coordinates of \( x \) so that this \( x^3 \) term has coefficient 1. We seek to “complete the cube” on the right-hand side. Given the coefficient of \( x^2y \) in \( Q \), there are specific coefficients for \( xy^2 \) and \( y^3 \) such that the right-hand side is a perfect cube. By making the change of coordinates

\[ z \mapsto \lambda x + \mu y + z, \]

with the appropriate choices of \( \lambda \) and \( \mu \), we produce the terms \( \lambda xy^2 \) and \( \mu y^3 \), which can be used to complete the cube. This produces

\[ y^2z = (x + by)^3. \]

With the final change of coordinates \( x^{\text{new}} = x + by \), we end with

\[ y^2z = x^3 \]

as desired.

Now consider the second case, \( xyz = Q(x, y) \). As before, \( Q \) must contain an \( x^3 \) and a \( y^3 \) term, or else it would violate irreducibility. First make scaling change
of coordinates of $x$ and $y$ so that the coefficients of $x^3$ and $y^3$ are 1. Then by making the change of coordinates $z \mapsto \lambda x + \mu y + z$, we once again complete the cube on the right-hand side (this time filling in the terms $x^2y$ and $xy^2$). This produces

$$xyz = (x + y)^3.$$ 

Now make the change of coordinates

$$\begin{align*}
x' &= x + y \\
y' &= x - y \\
z' &= -z/4
\end{align*} \quad \text{with inverse} \quad \begin{align*}
x &= (x' + y')/2 \\
y &= (x' - y')/2 \\
z &= -4z'
\end{align*}$$

Under these change of coordinates, the equation becomes

$$(y^2 - x^2)z = x^3,$$ 

or equivalently $y^2z = x^2(x + z)$ as desired.

(ii) Consider the case of the nodal cubic

$$f = x^2(x + z) - y^2z = 0.$$ 

Then

$$\frac{\partial f}{\partial x} = 3x^3 + 2xz, \quad \frac{\partial f}{\partial y} = -2yz, \quad \frac{\partial f}{\partial z} = x^2 - y^2.$$ 

If the second is to vanish, then $y = 0$ or $z = 0$. If $y = 0$, then from the $z$-partial we get $x = 0$, so we find the point $[0 : 0 : 1]$, which is indeed singular. If $z = 0$, then from the $x$-partial we get $x = 0$, then from the $z$-partial we get $z = 0$. But then $x = y = z = 0$, which is impossible. Therefore, the only singularity is $[0 : 0 : 1]$.

For the cuspidal cubic

$$f = x^3 - y^2z = 0,$$ 

the partials are

$$\frac{\partial f}{\partial x} = 3x^2, \quad \frac{\partial f}{\partial y} = -2yz, \quad \frac{\partial f}{\partial z} = -y^2.$$ 

If all are to vanish, then we must have $x = y = 0$, so we get the unique singular point $[0 : 0 : 1]$.

The reducible cubic given by

$$xyz = 0$$

has the three singularities $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. \qed
Problem 5. Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x, y, z]$. Consider the morphism
$$
\Phi : C \to \mathbb{P}^2, p \mapsto \left[ \frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p) \right].
$$
The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to $C$.

(i) Why is $\Phi$ a well-defined morphism? Find a geometric description of $\Phi$, independent of coordinate choices.

(ii) If $C$ is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic $C$ is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$. How does the morphism $\Phi$ change when we change coordinates?

(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

Answer: (i) If $p$ is a nonsingular point of $C$, $\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)$ can’t be zero simultaneously. Thus $\Phi$ is well-defined. The line $L_p$ with equation
$$
\frac{\partial f}{\partial x}(p)x + \frac{\partial f}{\partial y}(p)y + \frac{\partial f}{\partial z}(p)z = 0
$$
is the tangent line to $C$ at $p$.

(ii) From previous homeworks, we’ve learned to use linear coordinate changes to make a irreducible conic inti the form
$$
\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.
$$
If one of the $\lambda_i = 0$, the conic is reducible. So we can assume $\lambda_i \neq 0$ for all $i$. Then
$$
\Phi : C \to \mathbb{P}^2, [x : y : z] \mapsto [\lambda_1 x : \lambda_2 y : \lambda_3 z].
$$
Therefore, for points in the image $\Phi(C)$ we have
$$
X = \lambda_1 x, Y = \lambda_2 y, Z = \lambda_3 z, \text{ with } \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.
$$
Thus $\Phi(C)$ is the conic
$$
\frac{1}{\lambda_1} X^2 + \frac{1}{\lambda_2} Y^2 + \frac{1}{\lambda_3} Z^2 = 0.
$$
The geometric desprcription of $\Phi$ is to send a point $p$ to tangent lines of $C$ at $p$. This is independent of coordinates. Thus, the expression of $\Phi$ after coordinate changes is the same.

(iii) Let $C$ be an arbitrary conic. We claim that the dual of the dual of $C$ is $C$. Indeed, since the description of $\Phi$ is independent of coordinates, we may first assume that $C$ is of the form
$$
\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.
$$
We have seen above that the dual conic becomes
\[ \frac{1}{\lambda_1}X^2 + \frac{1}{\lambda_2}Y^2 + \frac{1}{\lambda_3}Z^2 = 0. \]
Dualizing once more, we recover the original conic \( C \).

Suppose \( L_i \) are 5 lines \( a_i x + b_i + c_i z = 0 \) such that the 5 points \( [a_i, b_i, c_i] \in \mathbb{P}^2 \) are in general position, as defined in Problem Set 4. Let \( D \) be the unique conic passing through the 5 points \( [a_i, b_i, c_i] \). Thus the dual conic of \( D \) is still a conic.

By definition, the dual conic of \( D \) in \( \mathbb{P}^2 \) is tangent to the five lines \( L_i \).

□

**Problem 6.** Consider the singular plane curves \( Z \) and \( W \) given by the equations
\[ y^2 - x^2(x + 1) = 0 \text{ and } xy = 0 \]
respectively.

(i) Explain briefly why \( Z \) and \( W \) are not isomorphic. Explain that \((0,0)\) is an ordinary double point for both of these curves. What are the tangent directions at \((0,0)\) for \( Z \) and \( W \)? Sketch (the real points of) \( Z \) and \( W \). Do \( Z \) and \( W \) look alike near the origin?

(ii) Show that there are formal power series
\[ \tilde{x} = f_1 + f_2 + f_3 + \ldots \text{ and } \quad \tilde{y} = g_1 + g_2 + g_3 + \ldots \]
in the variables \( x \) and \( y \) such that the equation of \( Z \) becomes
\[ \tilde{x}\tilde{y} = 0. \]

(iii) Explain briefly why any ordinary double point singularity in \( \mathbb{A}^2 \) is analytically equivalent to the node \( \tilde{x}\tilde{y} = 0 \).

**Answer:**

(i) First, \( Z \) and \( W \) are not isomorphic because \( Z \) is irreducible and \( W \) is reducible.

Now, \( Z \) is defined by \( y^2 - x^2(x + 1) = 0 \). The tangent lines can be found by considering the lowest degree terms \( y^2 - x^2 \). This factors as \((y - x)(y + x)\). So the 2 tangent lines are \( y - x = 0 \) and \( y + x = 0 \).

Similarly, \( W = (xy = 0) \) is the union of two lines. The tangent lines are just these two lines \( x = 0 \) and \( y = 0 \).

(ii) We hope to find \( f_i \) and \( g_i \) such that
\[ (f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1). \]
First, compare the degree 2 terms, then
\[ f_1g_1 = y^2 - x^2. \]
Hence, we can take

\[ f_1 = y - x \text{ and } g_1 = y + x. \]

Comparing the degree 3 terms we have

\[ -x^3 = (y - x)g_2 + (y + x)f_2. \]

The polynomials \( g_2 = x^2/2 \) and \( f_2 = -x^2/2 \) will work.

Suppose we have found \( f_i \) and \( g_i \) for \( 1 \leq i \leq d - 1 \) and

\[ (f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1) \]

up to degree \( d \). Comparing degree \( d + 1 \) terms, we have:

\[ f_1g_d + f_2g_{d-1} + \cdots + f dg_1 = 0. \]

Now only \( f_d \) and \( g_d \) are unknown, others are fixed, we can rearrange the equation:

\[ (y - x)g_d + (y + x)f_d = -(f_2g_{d-1} + \cdots + f_{d-1}g_2). \]

Notice that

\[ f_2g_{d-1} + \cdots + f_{d-1}g_2 \]

is a homogeneous polynomial of degree \( d + 1 \). Let

\[ -(f_2g_{d-1} + \cdots + f_{d-1}g_2) = ax^d + yR(x, y). \]

Isolating \( x^{d+1} \) and dividing the remaining term by \( y \) to obtain \( R \), then we need

\[ x(f_d - g_d) + y(f_d + g_d) = ax^{d+1} + yR(x, y) \]

This is possible by letting

\[ f_d = \frac{1}{2} \left( \frac{a}{2} x^d + R(x, y) \right) \text{ and } g_d = \frac{1}{2} \left( -\frac{a}{2} x^d + R(x, y) \right) \]

Therefore we can find \( f_d \) and \( g_d \) and

\[ (f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1) \]

up to degree \( d + 1 \). Inductively, there exist

\[ \tilde{x} = f_1 + f_2 + f_3 + \ldots \]

\[ \tilde{y} = g_1 + g_2 + g_3 + \ldots \]

such that

\[ (f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1). \]
(iii) Suppose \( C = (H = 0) \) is a curve which has an ordinary double point, we can change coordinates to assume that the singularity is at the origin. Because \( C \) has a double point at the origin,

\[
H(x, y) = H_2(x, y) + H_3(x, y) + \cdots \quad \text{deg}\, H_i = k
\]

where \( H_2 \) is a homogeneous polynomial of degree 2, with distinct factors

\[
H_2 = (ax + by)(cx + dy).
\]

Change coordinates so that

\[
x' = ax + by, \quad y' = cx + dy.
\]

In the new coordinates,

\[
H = x'y' + H'_3 + H'_4 + \ldots.
\]

By the same method as in (ii), we inductively find

\[
\hat{x} = x' + f_2 + f_3 + \ldots
\]

\[
\hat{y} = y' + g_2 + g_3 + \ldots
\]

such that

\[
H = \hat{x}\hat{y}.
\]