Math 203, Problem Set 1. Due Friday October 6.

For this problem set, you may assume that the ground field is algebraically closed.

1. Show that the Zariski topology on \( \mathbb{A}^2 \) is not the product of the Zariski topologies on \( \mathbb{A}^1 \times \mathbb{A}^1 \).

2. A topological space \( X \) is said to be Noetherian if it satisfies the ascending chain condition on open sets, i.e. any ascending chain of open sets eventually stabilizes.
   (i) Check that any subset \( Y \subset X \) of a Noetherian space is also Noetherian in the subspace topology.
   (ii) Show that \( \mathbb{A}^n \) is Noetherian in the Zariski topology. Conclude that any affine algebraic set is Noetherian.
   (iii) (Optional) Show that a Noetherian space is quasi-compact i.e., show that any open cover of a Noetherian space has a finite subcover.

3. Let \( \mathbb{A}^3 \) be the 3-dimensional affine space with coordinates \( x, y, z \). Find the ideals of the following algebraic sets:
   (i) The union of the \( (x, y) \)-plane with the \( z \)-axis.
   (ii) The image of the map \( \mathbb{A}^1 \to \mathbb{A}^3 \) given by \( t \to (t, t^2, t^3) \). The image is called the twisted cubic curve.

4. Let \( f : \mathbb{A}^n \to \mathbb{A}^m \) be a polynomial map i.e. \( f(p) = (f_1(p), \ldots, f_m(p)) \) for \( p \in \mathbb{A}^n \), where \( f_1, \ldots, f_m \) are polynomials in \( n \) variables. Are the following true or false:
   (i) The image \( f(X) \subset \mathbb{A}^m \) of an affine algebraic set \( X \subset \mathbb{A}^n \) is an affine algebraic set.
   (ii) The inverse image \( f^{-1}(X) \subset \mathbb{A}^n \) of an affine algebraic set \( X \subset \mathbb{A}^m \) is an affine algebraic set.
   (iii) If \( X \subset \mathbb{A}^n \) is an affine algebraic set, then the graph \( \Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m} \) is an affine algebraic set.

5. Let \( X \) be the union of the three coordinate axes in \( \mathbb{A}^3 \). Determine generators for the ideal \( I(X) \). Show that \( I(X) \) cannot be generated by fewer than 3 elements.

   Remark: Note that \( X \) has dimension 1 even though it is cut out by 3 equations. We say that \( X \) is not a complete intersection.

   Hint: Argue by contradiction. It may help to examine \( I(X)^{(2)} \) the degree 2 homogeneous part of the ideal \( I(X) \) as a vector space over \( k \).

6. Let \( X_1, X_2 \) be affine algebraic sets in \( \mathbb{A}^n \). Assuming the Nullstellensatz if necessary, show that
   (i) \( I(X_1 \cup X_2) = I(X_1) \cap I(X_2) \),
   (ii) \( I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} \).

   Show by example that taking the radical in (ii) is in general necessary, i.e. find affine algebraic sets \( X_1, X_2 \) such that \( I(X_1 \cap X_2) \neq I(X_1) + I(X_2) \).