Math 203, Problem Set 3. Due Friday October 20.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

1. Let \( n \geq 2 \), and let \( S = \{a_1, \ldots, a_n\} \) be a finite set with \( n \) elements in \( \mathbb{A}^1 \).

(i) Show that the quasi-affine set \( \mathbb{A}^1 \setminus S \) is isomorphic to an affine set. For instance, you may take \( X \) to be the affine algebraic set given by the equations

\[
X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.
\]

(ii) Show that \( \mathbb{A}^1 \setminus S \) is not isomorphic to \( \mathbb{A}^1 \setminus \{0\} \) by proving that their rings of regular functions are not isomorphic.

**Hint:** Assume that \( \Phi : A(X) \to k[t, t^{-1}] \) is an isomorphism. Observe that \( X_i \) are invertible elements in \( A(X) \) for all \( 1 \leq i \leq n \). Show that their images must be invertible in \( k[t, t^{-1}] \). Prove that this implies that \( \Phi(X_i) = t^{m_i} \) for some integers \( m_i \). Derive a contradiction by comparing \( \Phi(X_0 - a_i) \) for different values of \( i \).

2. Let \( n \geq 2 \). Consider the affine algebraic sets in \( \mathbb{A}^2 \):

\[
Z_n = \mathbb{Z}(y^n - x^{n+1})
\]

and

\[
W_n = \mathbb{Z}(y^n - x^n(x + 1)).
\]

Show that \( Z_n \) and \( W_n \) are birational but not isomorphic.

(i) Show that

\[
f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1})
\]

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

\[
\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0, 0)\}.
\]

Similarly, show that

\[
g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t).
\]

is a morphism of affine algebraic sets. Find open subsets of \( \mathbb{A}^1 \) and \( W_n \) where \( g \) becomes an isomorphism.

(ii) Using (i), explain why \( Z_n \) and \( W_n \) are birational.

(iii) Assume that there exists an isomorphism

\[
h : Z_n \to W_n
\]

such that \( h((0, 0)) = (0, 0) \). Observe that this induces an isomorphism between the open sets

\[
Z_n \setminus \{(0, 0)\} \to W_n \setminus \{(0, 0)\}.
\]

Use part (i) and the previous problem to conclude this cannot be true if \( n \geq 2 \).

(iv) *(Optional.)* Repeat the argument above without the assumption that \( h \) sends the origin to itself. You may need to prove a stronger version of Problem 2.
3. (Rational functions on prevarieties.) Let $X$ be a prevariety. Consider pairs $(U, f)$ where $U$ is an open subset of $X$ and $f \in \mathcal{O}_X(U)$ a regular function on $U$. We call two such pairs $(U, f)$ and $(U', f')$ equivalent if there is a nonempty open subset $V \subset U \cap U'$ such that

$$f|_V = f'|_V.$$

(i) Show that this defines an equivalence relation.

(ii) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the field of rational functions on $X$ and denoted $K(X)$.

(iii) If $X$ is an affine variety, show that $K(X)$ is just the field of rational functions as defined in class.

4. (Quotients.) Taking quotients in algebraic geometry is subtle. We will explain how to take quotients by finite groups.

Let $X$ be an affine variety, and let $G$ be a finite group. Assume that $G$ acts on $X$ algebraically, i.e. that for every $g \in G$, we are given a morphism $g : X \to X$ (denoted by the same letter for simplicity of notation), such that $(gh)(p) = g(h(p))$

for all $g, h \in G$ and $p \in X$.

(i) Let $g \in G$ act on the coordinate rings $A(X)$ via

$$f \mapsto f^g \text{ with } f^g(p) = f(g(p)).$$

Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all $G$-invariant functions on $X$. Show that $A(X)^G$ is a finitely generated $k$-algebra.

(ii) By (i), there is an affine variety $Y$ with coordinate ring $A(X)^G$, together with a morphism

$$\pi : X \to Y$$

determined by the inclusion

$$A(X)^G \hookrightarrow A(X).$$

Show that $Y$ can be considered as the quotient of $X$ by $G$, denoted $X/G$, in the following sense: if $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(iii) Let

$$\mu_n = \left\{ \exp \left( \frac{2\pi ik}{n} \right), k \in \mathbb{Z} \right\}$$

be the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Describe $\mathbb{C}/\mu_n$ and $\mathbb{C}^2/\mu_n$ as affine algebraic sets.