Math 203, Problem Set 5. Due Friday Nov 3.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

1. (Intersections with hypersurfaces.)
   (i) Show that if \( X \) is a hypersurface in \( \mathbb{P}^n \) and \( Y \subset \mathbb{P}^n \) is a projective variety which is not a point, then the intersection of \( X \) and \( Y \) is non-empty. You may want to recall that \( \mathbb{P}^n \setminus X \) is affine.

   In particular, any two curves \( C = Z(f) \) and \( D = Z(g) \) in \( \mathbb{P}^2 \) (with \( f \) and \( g \) nonconstant homogeneous polynomials) intersect in at least one point.

   (ii) If \( X \) is a degree \( d \) hypersurface in \( \mathbb{P}^n \) and \( L \) is a line in \( \mathbb{P}^n \) not contained in \( X \), show that \( L \) and \( X \) intersect in at most \( d \) points. In fact, they intersect in exactly \( d \) points counted with multiplicity.

   \textbf{Hint:} Change coordinates so that \( L \) is cut out by the equations
   \[ X_1 = \ldots = X_{n-2} = 0. \]

2. (Rational varieties and their birational automorphisms.) The definition of birational isomorphisms given in class extends to the projective category. Two projective varieties \( X \) and \( Y \) are birational if there are rational maps
   \[ f : X \rightarrow Y, \quad g : Y \rightarrow X, \]
   which are rational inverses to each other. Just as in the affine case, a birational isomorphism \( f : X \rightarrow Y \) induces an isomorphism of the fields of rational functions
   \[ f^* : K(Y) \rightarrow K(X) \]
   and conversely.

   (i) We say that \( X \) is rational if \( X \) is birational to \( \mathbb{P}^n \) for some \( n \). Explain that if \( X \) is rational, then \( K(X) \cong k(t_1, \ldots, t_n) \).

   (ii) Show that \( \mathbb{P}^n \times \mathbb{P}^m \) is rational, by constructing an explicit birational isomorphism with \( \mathbb{P}^{n+m} \). Show that if \( X \) and \( Y \) are rational, then \( X \times Y \) is rational.

   \textbf{Remark:} It is very difficult to determine if a given variety is rational. We have seen that lines, conics in \( \mathbb{P}^2 \) are rational, while elliptic curves in \( \mathbb{P}^2 \) are not. Twisted cubics in \( \mathbb{P}^3 \) are rational. We will prove below that quadrics in \( \mathbb{P}^3 \) are rational. Smooth cubic surfaces in \( \mathbb{P}^3 \) also turn out to be rational. However, most varieties are not rational.

   (iii) Show that \( \mathbb{P}^2 \) is not isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) (but they are birationally isomorphic). You may want to find two curves in \( \mathbb{P}^1 \times \mathbb{P}^1 \) which do not intersect.
(iv) The group of automorphisms of the field of fractions in two variables \( k(x, y) \) is called the Cremona group. Explain that the elements of the Cremona group correspond to birational self-isomorphisms of \( \mathbb{P}^2 \). Explain that the Cremona involution

\[(x, y) \rightarrow (x^{-1}, y^{-1})\]

extends to an automorphism of \( k(x, y) \). What is the corresponding birational involution of \( \mathbb{P}^2 \)? Where is this birational automorphism regular?

(v) More generally, show that \( GL_2(\mathbb{Z}) \) is a subgroup of the Cremona group.

Remark: The Cremona group is not yet fully understood (especially when the number of indeterminates \( t_i \) is bigger than 2).

3. (Quadrics are rational.) We have seen in class that an irreducible projective quadric \( Q \) corresponds to a symmetric square matrix \( A \) such that

\[Q(x) = x^T A x \text{ for } x \in \mathbb{P}^n.\]

We say that the quadric \( Q \) is nondegenerate if the matrix \( A \) is non-degenerate.

(i) Show that a non-degenerate irreducible quadric \( Q \) in \( \mathbb{P}^3 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), hence \( Q \) is rational.

(ii) In general, show that any non-degenerate irreducible quadric \( Q \subset \mathbb{P}^{n+1} \) is birational to \( \mathbb{P}^n \).

Hint: Pick \( p \in Q \) and assume after a change of coordinates that \( p = [1 : 0 : \ldots : 0] \). Consider the linear hyperplane

\[H = \{ X_0 = 0 \}.\]

For any \( q \in Q \), define \( f(q) \) to be the point of intersection of the line \( pq \) with \( H \). Show that

\[f : Q \longrightarrow H\]

is a birational isomorphism. This rational map is called the linear projection from \( p \).

4. (Elliptic curves and intersection of quadrics.)

In general, the intersection of two quadrics in \( \mathbb{P}^3 \) is an elliptic curve. Check this for the following two quadrics in \( \mathbb{P}^3 \):

\[Q_1 = Z(xw - yz), Q_2 = (yw - (x - z)(x - \lambda z)).\]

More precisely, show that the intersection of the two quadrics is isomorphic to the elliptic curve \( \overline{E}_\lambda \subset \mathbb{P}^2 \).

This problem may require a bit of patience.
5. (Twisted curves and complete intersections.)

A variety $Y$ of dimension $r$ in $\mathbb{P}^n$ is a strict complete intersection if the ideal $I(Y)$ can be generated by $n - r$ elements. $Y$ is a set-theoretic complete intersection if $Y$ can be written as the intersection of $n - r$ hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.
(ii) Show that the twisted cubic $T$ in $\mathbb{P}^3$ can be written as the set-theoretic intersection of the quadric and the cubic

$$Q = \mathcal{Z}(y^2 - xz), C = \mathcal{Z}(z^3 + xw^2 - 2yzw).$$

In particular, $T$ is a set theoretic complete intersection.
(iii) Show that $T$ is the intersections of the three quadrics

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xw - yz), Q_3 = \mathcal{Z}(yw - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic. In particular $I(T)$ contains three elements of degree 2.
(iv) Possibly using (iii), explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence $T$ is not a strict complete intersection.

Remark: It is an unsolved problem to show that every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic complete intersection. This is called the Hartshorne conjecture.

6. (Joins.) Let $\mathbb{G}(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Show that:

(i) The set $\{(L,P) : P \in L\} \subset \mathbb{G}(1,n) \times \mathbb{P}^n$ is closed.
(ii) If $Z \subset \mathbb{G}(1,n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.
(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting $X$ and $Y$ is a closed subset of $\mathbb{P}^n$. It is called the join $J(X,Y)$ of $X$ and $Y$. 