Math 203, Problem Set 6. Due Friday, November 10.

1. *(Semicontinuity of fiber dimension.)* If \( f : X \to Y \) is a surjective morphism of varieties, then the set
\[
Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \}
\]
is closed.

2. *(Criterion for irreducibility.)* Assume that \( f : X \to Y \) is a surjective morphism of projective algebraic sets such that \( Y \) is irreducible and all fibers of \( f \) are irreducible of the same dimension. Show that \( X \) is irreducible as well.

3. *(Intersections in projective space.)* Let \( X \) and \( Y \) be two subvarieties of \( \mathbb{P}^n \). Show that if \( \dim X + \dim Y \geq n \), then \( X \cap Y \) is not empty.  

*Hint:* Let \( H_1, H_2 \) be two disjoint linear subspaces of dimension \( n \) in \( \mathbb{P}^{2n+1} \), and consider \( X \subset H_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1} \) and \( Y \subset H_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1} \) as subvarieties of \( \mathbb{P}^{2n+1} \). Show that the join \( J(X,Y) \) in \( \mathbb{P}^{2n+1} \) has dimension \( \dim X + \dim Y + 1 \). Then construct \( X \cap Y \) as a suitable intersection of \( J(X,Y) \) with \( n + 1 \) hyperplanes.

4. *(Lines on hypersurfaces.)*

(i) Let \( d > 2n - 3 \). Show that a general degree \( d \) hypersurface in \( \mathbb{P}^n \) contains no lines. (That is, show that at least one such hypersurface contains no lines. This implies that in fact a random such hypersurface contains no lines.)

*Hint:* View each hypersurface \( X \) as point in \( \mathbb{P}^N \) for \( N = \binom{n+d}{d} - 1 \) by recording the coefficients of the defining equation. Consider the incidence correspondence
\[
J = \{ (L, X) : L \subset X \} \hookrightarrow \mathbb{G}(1, n) \times \mathbb{P}^N.
\]
Calculate the the dimension of fibers under the projections to the two factors. Conclude that there is a nonempty open set in \( \mathbb{P}^N \) corresponding to hypersurfaces without lines.

(ii) By (i), a general degree \( d > 3 \) surface in \( \mathbb{P}^3 \) contains no lines.

We had seen that for \( d = 2 \), all nondegenerate quadrics in \( \mathbb{P}^3 \) are isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), so they carry plenty of lines.

We will see later that for \( d = 3 \), all cubic surfaces in \( \mathbb{P}^3 \) contain lines and that smooth cubic surfaces have exactly 27 lines.

Here, we consider the case \( d = 4 \). Let \( f \) be a degree 4 homogeneous polynomial in 4 variables and let \( Z_f \) be the quartic surface \( f = 0 \) in \( \mathbb{P}^3 \). Show that there is a single polynomial \( \Phi \) in the coefficients of \( f \) such that
\[
\Phi(f) = 0 \iff Z_f \subset \mathbb{P}^3 \text{ contains a line.}
\]

*Remark:* This polynomial is a homogeneous polynomial of degree 320 in 35 variables, so you should expect it to have approximately \( 10^{47} \) nonzero coefficients.