Math 203, Problem Set 7. Due Friday, December 1.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

This problem set consists of 7 questions selected out of the following 9.

Solve the following two problems, but only hand in either Problem 1 or Problem 2.

1. (Hyperelliptic curves.) Let \( a_1, \ldots, a_{2g+1} \) be pairwise distinct constants. Find the singularities of the projective hyperelliptic curve of genus \( g \):

\[
y^2z^{2g-1} = (x - a_1z) \cdots (x - a_{2g+1}z).
\]

Remark: When \( g = 1 \), we get the elliptic curve \( E_\lambda \) already encountered before, by setting \( a_1 = 0, a_2 = 1, a_3 = \lambda \).

2. (Dual conics.) Let \( C \subset \mathbb{P}^2 \) be a non-singular curve, given as the zero locus of a homogeneous polynomial \( f \in k[x,y,z] \). Consider the morphism

\[
\Phi : C \to \mathbb{P}^2, p \mapsto \left[ \frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p) \right].
\]

The image \( \Phi(C) \subset \mathbb{P}^2 \) is called the dual curve to \( C \).

(i) Why is \( \Phi \) a well-defined morphism? Find a geometric description of \( \Phi \), independent of coordinates.

(ii) If \( C \) is an irreducible conic, prove that its dual \( \Phi(C) \) is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic \( C \) is \( ax^2 + by^2 + cz^2 = 0 \). What is \( \Phi(C) \)?

(iii) For any five lines in \( \mathbb{P}^2 \) in general position (what does this mean?) show that there is a unique conic in \( \mathbb{P}^2 \) that is tangent to these five lines.

Solve and hand in Problems 3, 4, 5, 6.

3. (Singularities of hypersurfaces.) Show that a general hypersurface of degree \( d \) in \( \mathbb{P}^n \) is non-singular:

(i) For any hypersurface \( Z(f) \subset \mathbb{P}^n \) of degree \( d \), view the coefficients of \( f \) as a point \( p_f \) in a large dimensional projective space \( \mathbb{P}^N \) (This projective space is called the moduli space of degree \( d \) hypersurfaces). Let

\[
X = \{(f,p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f \}.
\]

Show that \( X \) is a projective algebraic set in \( \mathbb{P}^N \times \mathbb{P}^n \).

(ii) Conclude that the image \( \pi(X) \) of \( X \) under the projection onto \( \mathbb{P}^N \) is a projective algebraic set. What is \( \pi(X) \)? Conclude that the subset of \( \mathbb{P}^N \) corresponding to smooth hypersurfaces is open and nonempty.

Remark: This will prove that the hypersurface is singular provided that the coefficients of \( f \) satisfy certain polynomial relations. Therefore, if you pick \( f \)
randomly, these polynomial relations will most likely not be satisfied and your hypersurface is non-singular. This is the explanation of the word *general*.

4. (*Analytic singularities.*) Consider the singular plane curves $Z$ and $W$ given by the equations

$$y^2 - x^2(x + 1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain that $(0, 0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0, 0)$ for $Z$ and $W$? Sketch (the real points of) $Z$ and $W$. Do $Z$ and $W$ look alike near the origin?

(ii) Show that there are *formal power series*

$$\tilde{x} = f_1 + f_2 + f_3 + \ldots \text{ and } \tilde{y} = g_1 + g_2 + g_3 \ldots$$

in the variables $x$ and $y$ such that the equation of $Z$ becomes

$$\tilde{x}\tilde{y} = 0.$$ 

*Hint:* Construct the degree $i$ homogeneous parts $f_i$ and $g_i$ inductively. Show you can pick

$$f_1 = y - x, g_1 = x + y.$$

Next, you would need

$$f_2(x + y) + g_2(y - x) = -x^3.$$ 

Why can you construct $f_2$ and $g_2$? Continue in this fashion.

*Remark:* If we work over an arbitrary field $k$ it doesn’t make sense to ask if the power series $\tilde{x}$ and $\tilde{y}$ converge, hence the terminology *formal power series*. Convergence may be arranged if you work over the complex numbers, but you don’t have to prove it.

*Remark:* It turns out the assignment

$$(x, y) \rightarrow (\tilde{x}, \tilde{y})$$

is invertible e.g. you can solve for $x, y$ in terms of formal power series in $\tilde{x}, \tilde{y}$. In fact, this statement is generally true about any power series

$$\tilde{x} = ax + by + \ldots, \tilde{y} = cx + dy + \ldots$$

provided that $ad - bc \neq 0$. Therefore, the assignment

$$(x, y) \rightarrow (\tilde{x}, \tilde{y})$$

is a *formal* change of coordinates, establishing a *formal isomorphism* between $Z$ and $W$. We say that $Z$ and $W$ are *analytically equivalent*.

*Remark:* Over the complex numbers, convergence may be arranged near the origin, if $x, y$ are small, and thus the word *formal* may be replaced by *local analytic isomorphism* near the origin.
(iii) Explain briefly why any ordinary double point singularity in $A^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

Remark: It can be shown that any double point is analytically equivalent to the singularity $\tilde{y}^2 = \tilde{x}^r$, for some $r$. The case $r = 2$ corresponds to the case which concerned us above.

5. (Normal varieties.) Show that the quadric $x^2 + y^2 + z^2 = 0$ in $A^3$ is normal.

Hint: Consider $\alpha \in K(X)$. Using that $z^2 = -x^2 - y^2$, show that $\alpha = u + zv$ for $u, v \in k(x, y)$. Assume that $\alpha$ is integral. Show that the minimal polynomial of $\alpha$ over $k(x, y)$ is $T^2 - 2uT + (u^2 + v^2(x^2 + y^2)) = 0$. Use that its coefficients must be in $k[x, y]$ (why?) and conclude that $u, v$ are polynomials. Conclude that $\alpha \in A(X)$.

6. (Resolving curve singularities.) Resolve the following $A_k$ plane curve singularity by subsequent blow-ups

$$y^2 - x^{k+1} = 0.$$

Remark: We have the following terminology on isolated “simple” singularities of hypersurfaces in $A^{n+2}$:

- type $A_k$: $x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $D_k$: $x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_6$: $x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_7$: $x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type $E_8$: $x^5 + y^3 + z_2^2 + \ldots + z_n^2 = 0$.

(The names suggest a connection with the Weyl groups of type $A, D, E$.)

Solve the following two problems, but only hand in either Problem 7 or Problem 8.

7. (Tangent cones.) Let $X \subset A^n$ be an affine variety and let $p \in X$. Let $\mathfrak{m}$ be the maximal ideal of $O_{X,p}$. Show that the coordinate ring $A(C_{X,p})$ of the tangent cone of $X$ at $p$ is isomorphic to the graded algebra $\bigoplus_{k \geq 0} \mathfrak{m}^k / \mathfrak{m}^{k+1}$.

Hint: Let $i \subset k[x_1, \ldots, x_n]$ be the ideal of $X$. Show that

$$k[x_1, \ldots, x_n]/i^m \to \bigoplus_{k \geq 0} \mathfrak{m}^k / \mathfrak{m}^{k+1}$$

given by $f \mapsto f^{(k)}|_X$ is an isomorphism.

8. (Exceptional hypersurface.) Consider the blowup of the affine variety $X \subset A^n$ at $p \in X$. Show that the exceptional hypersurface is the projectivization of the tangent cone

$$E \cong \mathbb{P}(C_{X,p}).$$

You may want to generalize the argument we had in class for plane curves.
Solve and hand in the following problem.

9. (Cremona transformations.) Consider the Cremona birational automorphism of \( \mathbb{P}^2 \) given by

\[ C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1]. \]

Let \( \widetilde{\mathbb{P}}^2 \) be the blowup of \( \mathbb{P}^2 \) at the three points \( P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0] \) and \( P_3 = [0 : 0 : 1] \) where \( C \) is undefined. Show that

(i) Show that \( C \) extends to an isomorphism \( \widetilde{C} : \widetilde{\mathbb{P}}^2 \rightarrow \widetilde{\mathbb{P}}^2. \)

(ii) Let \( E_1, E_2, E_3 \) be the exceptional lines for the blowup, and let \( L_{ij} \) be the strict transform of the line through \( P_i \) and \( P_j \). Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under \( \widetilde{C} \)?

*Hint:* Show that the equations of the blowup \( \widetilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2 \) are given by

\[ x_0y_0 = x_1y_1 = x_2y_2. \]

It may help to find the equations of the exceptional lines \( E_1, E_2, E_3 \) and those of the strict transforms \( L_{23}, L_{12}, L_{13} \).