Math 203, Problem Set 8. Due Friday, December 8.

1. **(Degree of the Segre embedding.)** Show that the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ has degree $\binom{n+m}{n}$.

2. **(Arithmetic genus.)** Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert polynomial $\chi_X$. Define the arithmetic genus of $X$ to be
   
   $$p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1).$$

   (i) Show that the genus of $\mathbb{P}^n$ is zero.
   (ii) If $X$ is a hypersurface of degree $d$ in $\mathbb{P}^n$, show that $p_a(X) = \binom{d-1}{n}$. In particular, a cubic in $\mathbb{P}^2$ has genus 1.
   (iii) If $X$ is a complete intersection of two surfaces of degree $a$ and $b$ in $\mathbb{P}^3$ then
   
   $$p_a(X) = \frac{1}{2}ab(a + b - 4) + 1.$$

   In particular, intersection of two quadrics in $\mathbb{P}^3$ has genus 1.

   **Remark:** To compare (ii) and (iii), recall that a cubic in $\mathbb{P}^2$ is isomorphic to an intersection of two quadrics in $\mathbb{P}^3$ as shown in a previous homework.

3. **(Enumerative geometry of lines.)** Given four general lines in $\mathbb{P}^3$, show that there are exactly 2 lines which intersect all four of them.

   **Hint:** Recall that the Grassmannian $G(1, 3)$ is a quadric in $\mathbb{P}^5$ via the Plücker embedding.

   **Remark:** The number of lines in $\mathbb{P}^n$ which intersect $2(n - 1)$ fixed general codimension 2 linear hyperplanes equals the Catalan number
   
   $$C_n = \frac{1}{n} \binom{2n - 2}{n - 1}.$$

4. **(Varieties of minimal degree.)** Let $X$ be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree $d$ and codimension $c$ in $\mathbb{P}^n$.
   (i) (Intersecting $X$ with hyperplanes to cut down the dimension), show inductively that
   
   $$d \geq c + 1.$$

   (ii) The Del Pezzo-Bertini theorem classifies all varieties for which equality holds in (i).
   Here, verify that equality holds for rational normal curves in $\mathbb{P}^n$, and for the image $v(\mathbb{P}^2)$ of the Veronese embedding
   
   $$v : \mathbb{P}^2 \to \mathbb{P}^5.$$

   (iii) Can you classify the varieties of degree 2?