Math 203A - Solution Set 4

**Problem 1.**

(i) Show that every isomorphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ is of the form $f(x) = ax + b$.

(ii) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in k$, where $x$ is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

(iii) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

**Answer:**

(i) Clearly, a morphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ must be a polynomial map. If $f$ has degree $d$, the equation $f(x) = 0$ will have $d$ roots. Therefore, $d = 1$ hence $f(x) = ax + b$.

(ii) Let $f : \mathbb{P}^1 \to \mathbb{P}^1$. If $f(\infty) = \infty$, then $f$ maps $\mathbb{A}^1 \to \mathbb{A}^1$ and must have the form $f(x) = ax + b$. Otherwise, if $f(\infty) = \lambda$, let $g(x) = \frac{1}{x-\lambda}$. Then $g \circ f$ is an isomorphism which maps $\infty$ to itself hence it must have the form

$$g \circ f = cx + d.$$

This shows

$$f(x) = \lambda + \frac{1}{cx + d} = \frac{ax + b}{cx + d}$$

for $a = c\lambda, b = d\lambda + 1$.

(iii) By transitivity, it suffices to assume $P_1 = 0, P_2 = 1, P_3 = \infty$. Also, let $Q_0 = \lambda, Q_1 = \mu, Q_2 = \nu$. Assume that none of the greek letters is $\infty$. We require

$$\frac{b}{d} = \lambda, \frac{a+b}{c+d} = \mu, \frac{a}{c} = \nu.$$

We may take

$$a = \nu(\mu - \lambda), \quad b = \lambda(\nu - \mu), \quad c = \mu - \lambda, \quad d = \nu - \mu.$$ 

If one of the greek letters is $\infty$, say $\nu = \infty$, and $\mu, \lambda \neq 0$, we may apply the transformation $x \to \frac{1}{x}$ to reduce to the case we have already studied.

If by contrast, $\lambda = 0$, then the automorphism $f(x) = \mu x$ sends the $P$’s to the $Q$’s. If $\mu = 0$, then we may take

$$f(x) = -\lambda(x - 1).$$

To prove uniqueness, assume that two morphisms $f_1, f_2$ have been constructed.

The inverse $f = f_1 \circ f_2^{-1}$ has 3 fixed points $P_1, P_2, P_3$. We may assume that these
fixed points are 0, 1, ∞. Since
\[ f(x) = \frac{ax + b}{cx + d} \]
and
\[ f(0) = 0, f(1) = 1, f(\infty) = \infty \]
we conclude
\[ b = 0, a + b = c + d, c = 0 \implies f = \text{id} \implies f_1 = f_2. \]
\[ \square \]

**Problem 2.** Prove the following facts about lines and conics in projective plane:

(i) For any line \( L \subset \mathbb{P}^2 \), there is a bijective morphism
\[ f : \mathbb{P}^1 \to L. \]

(ii) For any irreducible conic \( C \subset \mathbb{P}^2 \), there is a bijective morphism
\[ f : \mathbb{P}^1 \to C. \]

You may wish to change coordinates so that your conic has a convenient expression.

(iii) Consider the elliptic curve \( E_\lambda \subset \mathbb{P}^2 : \)
\[ y^2z = x(x-z)(x-\lambda z). \]
Show that there are no nonconstant morphisms
\[ \mathbb{P}^1 \to E_\lambda \subset \mathbb{P}^2. \]

Therefore, elliptic curves are not rational curves.

**Answer:**

(i) Let \( L \) be the line \( \alpha x + \beta y + \gamma z = 0 \). We may assume that \( \gamma \neq 0 \), eventually relabeling the coordinates if necessary. Set
\[ f : \mathbb{P}^1 \to L, \ [x : y] \to \left[ x : y : -\frac{\alpha}{\gamma}x - \frac{\beta}{\gamma}y \right], \]
and
\[ g : L \to \mathbb{P}^1 \ [x : y : z] \to [x : y]. \]

It is easy to see that both \( f \) and \( g \) are well defined inverse isomorphisms.

(ii) Changing coordinates, we may assume the conic \( C \) is given by the equation
\[ xz = y^2. \]

Set
\[ f : \mathbb{P}^1 \to C, \ [s : t] \to [s^2 : st : t^2]. \]
It is easy to check that
\[ g : C \to \mathbb{P}^1, g([x : y : z]) = \begin{cases} [x : y] & \text{if } (x, y) \neq (0, 0) \\ [y : z] & \text{if } (y, z) \neq (0, 0) \end{cases} \]
is an inverse morphism of \( f \).

(iii) Assume that
\[ F : \mathbb{P}^1 \to E_\lambda \subset \mathbb{P}^2 \]
is a morphism. Let \( E_\lambda \) be the affine piece where \( z = 1 \). Note that \( E_\lambda \setminus E_\lambda = \{[0 : 1 : 0]\} \). The preimage \( F^{-1}([0 : 1 : 0]) \) is a closed subset of \( \mathbb{P}^1 \) so it is a finite set \( S \). Restricting to \( E_\lambda \), we obtain a morphism \( \tilde{F} : \mathbb{P}^1 \setminus S \to E_\lambda \). In other words, there is a rational map \( f : \mathbb{A}^1 \dashrightarrow E_\lambda \) which as shown in class must be constant. This implies \( F \) is constant.

\[ \square \]

**Problem 3.**

(i) Four points in \( \mathbb{P}^2 \) are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if \( p_1, \ldots, p_4 \) are points in general position, there exists a linear change of coordinates
\[ T : \mathbb{P}^2 \to \mathbb{P}^2 \]
with
\[ T([1 : 0 : 0]) = p_1, \ T([0 : 1 : 0]) = p_2, \ T([0 : 0 : 1]) = p_3, \ T([1 : 1 : 1]) = p_4. \]

(ii) Given five distinct points in \( \mathbb{P}^2 \), no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part (i) to assume that four of the points are \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]\).

(iii) Deduce that two distinct irreducible conics in \( \mathbb{P}^2 \) cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

**Answer:**

(i) Let \( p_i = [a_i : b_i : c_i] \) for \( 1 \leq i \leq 4 \). Define
\[ A = \begin{pmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \beta a_2 & \beta b_2 & \beta c_2 \\ \gamma a_3 & \gamma b_3 & \gamma c_3 \end{pmatrix}, \]
where \( \alpha, \beta, \gamma \) will be specified later. In fact, we will require that \( \alpha, \beta, \gamma \) solve the system
\[ \begin{align*}
\alpha a_1 + \beta b_1 + \gamma c_1 &= a_4, \\
\alpha a_2 + \beta b_2 + \gamma c_2 &= b_4, \\
\alpha a_3 + \beta b_3 + \gamma c_3 &= c_4.
\end{align*} \]
A solution exists since the matrix of coefficients

\[ B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \]

is invertible. Indeed, the rows of \( B \) are independent. Otherwise, a nontrivial linear relation between the rows would give a line on which the points \( p_1, p_2, p_3 \) lie. Thus \( B \) is invertible. Now, the system above has the solution

\[ \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = B^{-1} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}. \]

Note that the same argument shows that \( A \) is invertible. Let \( S : \mathbb{P}^2 \to \mathbb{P}^2 \) be the linear transformation defined by \( A \). Then \( S \) is invertible. A direct computation shows

\[ S([1 : 0 : 0]) = p_1, S([0 : 1 : 0]) = p_2, S([0 : 0 : 1]) = p_3, S([1 : 1 : 1]) = p_4. \]

The proof is completed letting \( T \) be the inverse of \( S \).

(ii) After a linear change of coordinates, we may assume that the five points are \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1] \) and \([u : v : w] \). The equation \( f(x, y, z) \) of any conic passing through the first three points can’t contain \( x^2, y^2, z^2 \), so

\[ f(x, y, z) = ayz + bxz + cxy. \]

The remaining two points impose the conditions

\[ a + b + c = avw + bwu + cuv = 0. \]

Letting

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ vw & uw & uv \end{pmatrix}, \]

we see that \( (a, b, c) \) must be in the null space of \( A \). The rank of \( A \) is 2 (the rank cannot be 1 since then the rows would be proportional, hence \( u = v = w \) which is not allowed). Therefore, the null space of this matrix is one dimensional, hence the conic passing through the 5 points is unique. The conic cannot be reducible since then it would be union of two lines. One of the lines would have to contain 3 points but that contradicts the general position assumption.

(iii) We first show that a line and a conic intersect in at most 2 points. Indeed, changing coordinates, we may assume that the conic \( C \) is

\[ xz = y^2. \]
Let $L$ be the line

$$\alpha x + \beta y + \gamma z = 0.$$ 

If $\beta = 0$, it is clear that $C$ and $L$ intersect at the points

$$[-\gamma : \pm \sqrt{\alpha\gamma} : \alpha].$$

If $\beta \neq 0$, then

$$y = -\frac{\alpha}{\beta} x - \frac{\gamma}{\beta} z$$

which gives

$$xz = \frac{1}{\beta^2}(\alpha x + \gamma z)^2.$$ 

If $x = 0$, then $y = z = 0$ which is not allowed. Therefore, we may assume $x \neq 0$. Dividing by $x^2$ we obtain the quadratic equation in $\frac{z}{x}$:

$$\frac{\gamma^2}{\beta^2} \left( \frac{z}{x} \right)^2 + \left( \frac{2\alpha\gamma}{\beta^2} - 1 \right) \frac{z}{x} + \frac{\alpha^2}{\beta^2} = 0.$$ 

Letting $\lambda_1, \lambda_2$ be the solutions of this equation, we see that the intersection points are

$$\left[ 1 : \frac{-\alpha}{\beta} - \frac{\gamma}{\beta} \lambda_i : \lambda_i \right].$$

By the above, a conic and a line cannot intersect in 3 points. Therefore, any 5 points on an irreducible conic are in general position. Now, the claim is evident by (ii).

\[\square\]

**Problem 4.** We will make the space of all lines in $\mathbb{P}^n$ into a projective variety. We define a set-theoretic map

$$\phi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with

$$N = \binom{n+1}{2} - 1$$

as follows. For every line $L \subset \mathbb{P}^n$, choose two distinct points

$$P = (a_0 \ldots a_n) \text{ and } Q = (b_0 \ldots b_n)$$

on $L$ and define $\phi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the maximal minors of the matrix

$$\begin{pmatrix} a_0 & \ldots & a_n \\ b_0 & \ldots & b_n \end{pmatrix}$$

in any fixed order. Show that:

(i) The map $\phi$ is well-defined and injective. The map $\phi$ is called the Plucker embedding.
(ii) The image of $\phi$ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. You may want to recall the Gaussian algorithm which brings almost any matrix as above into the form

$$
\begin{pmatrix}
1 & 0 & a'_2 & \ldots & a'_n \\
0 & 1 & b'_2 & \ldots & b'_n
\end{pmatrix},
$$

(iii) Show that $G(1, 1)$ is a point, $G(1, 2) = \mathbb{P}^2$, and $G(1, 3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.

**Answer:**

(i) Let $e_0, \ldots, e_n$ be the standard basis of the vector space $V = \mathbb{K}^{n+1}$. A line $L$ corresponds to a 2-dimensional subspace in $\mathbb{K}^{n+1}$, also denoted by $L$. We claim that the map $\phi$ can be described as follows. Picking a basis $v, w$ for the subspace,

$$
\phi(L) = [v \wedge w] \in \mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^N.
$$

Indeed, if $P = (a_0 : \ldots : a_n)$ and $Q = (b_0 : \ldots : b_n)$ are two distinct points, then we may take

$$
v = \sum a_i e_i, w = \sum b_i e_i.
$$

Thus

$$
v \wedge w = \sum_{i,j} a_i b_j e_i \wedge e_j = \sum_{i<j} (a_i b_j - a_j b_i) e_i \wedge e_j.
$$

Therefore, in the basis $e_i \wedge e_j$, the coordinates are the $2 \times 2$-minors of the matrix in (i). This shows that $\phi$ is well-defined.

To check injectivity, let $L'$ be another line corresponding to a 2-dimensional subspace. If $L' \cap L = 0$, then pick a basis $v_0, v_1, v_2, v_3$ for $L \oplus L'$ with

$$
v_0, v_1 \in L, v_2, v_3 \in L',
$$

and extend it to a basis $v_0, \ldots, v_n$ of $V = \mathbb{K}^{n+1}$. Thus $v_i \wedge v_j$ for $i < j$ is a basis for $\Lambda^2 V$. In particular, $v_0 \wedge v_1$ and $v_2 \wedge v_3$ are not multiples, or equivalently

$$
\phi(L) \neq \phi(L').
$$

The same argument applies if $L$ and $L'$ have a 1 dimensional intersection.

(ii) To show projectivity, we will prove that

$$
\omega \in \Lambda^2 V \text{ splits as } \omega = v \wedge w \text{ if and only if } \omega \wedge \omega = 0.
$$

In particular, if

$$
\omega = \sum \omega_{ij} e_i \wedge e_j,
$$

then

$$
\omega \wedge \omega = \sum_{i<j, k<l} \omega_{ij} \omega_{kl} e_i \wedge e_j \wedge e_k \wedge e_l = \sum_{i<j<k<l} (\omega_{ij} \omega_{kl} - \omega_{ik} \omega_{jl} + \omega_{il} \omega_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l
$$
so the image of \( \phi \) is cut by the quadrics

\[
\omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk} = 0.
\]

We now prove the claim. It is clear that if \( \omega = v \wedge w \) then \( \omega \wedge \omega = 0 \). Conversely, we will induct on \( n \), the base case \( n = 2 \) being clear. Let us write

\[
\omega = e_0 \wedge \eta + \omega'
\]

where \( \omega', \eta \) do not contain the vector \( e_0 \). Thus

\[
0 = \omega \wedge \omega = 2e_0 \wedge \eta \wedge \omega' + \omega' \wedge \omega'.
\]

This implies that

\[
\omega' \wedge \omega' = 0
\]

hence by induction

\[
\omega' = v \wedge w,
\]

with \( v, w \) being in the subspace spanned by \( e_1, \ldots, e_n \). Also, we know

\[
e_0 \wedge \eta \wedge \omega' = 0 \quad \implies \quad \eta \wedge v \wedge w = 0.
\]

This shows that \( \eta \) cannot be independent of \( v, w \) hence

\[
\eta = av + bw.
\]

Collecting terms we find

\[
\omega = e_0 \wedge (av + bw) + v \wedge w = (v + be_0) \wedge (w + ae_0)
\]

as claimed.

For the last part, note that one of the coordinates of \( \phi(L) \) must be non-zero. Without loss of generality let us assume it is the coordinate corresponding to \( e_0 \wedge e_1 \). This means that the first \( 2 \times 2 \) minor of the matrix

\[
\begin{pmatrix}
a_0 & \cdots & a_n \\
b_0 & \cdots & b_n
\end{pmatrix}
\]

is non-zero. The Gaussian algorithm brings this matrix into the form

\[
\begin{pmatrix}
1 & 0 & a'_2 & \ldots & a'_n \\
0 & 1 & b'_2 & \ldots & b'_n
\end{pmatrix},
\]

The association

\[
L \to (a'_2, \ldots, a'_n, b'_2, \ldots, b'_n) \in \mathbb{A}^{2(n-1)}
\]

shows how to cover the image of \( \phi \) by affine opens isomorphic to \( \mathbb{A}^{2(n-1)} \).

Finally, to show that the Grassmannian is irreducible, one proves first that if \( X \) is a projective algebraic set that can be covered by open irreducible subsets
$U_i$ such that $U_i \cap U_j \neq \emptyset$, then $X$ is irreducible. This is clearly the case in our setup for the affine charts we constructed.

To prove the claim, assume that $V, W$ are nonempty open sets in $X$. We show $V, W$ intersect. Indeed,

$$U_i = (V \cap U_i) \cup (W \cap U_i)$$

are then open sets in $U_i$. If $V \cap U_i$ and $W \cap U_i$ are nonempty, since $U_i$ is irreducible, we must have

$$(V \cap U_i) \cap (W \cap U_i) \neq \emptyset \implies V \cap W \neq \emptyset.$$  

We are left to analyze the case when for each $i$ either $V \cap U_i$ or $W \cap U_i$ is empty. WLOG we may assume there exists $i$ such that $V \cap U_i$ is empty. Since

$$V = \cup_j (V \cap U_j)$$

it follows that there exists $j$ such that $V \cap U_j \neq \emptyset$. But then for any $i$, $V \cap U_j$ and $U_i \cap U_j$ are nonempty open subsets of $U_j$ which is irreducible. Therefore, $U_i \cap U_j$ and $V \cap U_j$ must intersect or in other words $V \cap U_i$ must intersect as well, contradiction.

(iii) All these statements are particular cases of what we proved in part (i). For instance, to see that $G(1, 3)$ is a quadric in $\mathbb{P}^5$ given by

$$x_0x_5 - x_1x_4 + x_2x_3 = 0.$$  

□

**Problem 5.** (Introduction to moduli theory.) Show that for any 3 lines $L_1, L_2, L_3$ in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric $Q$ and not on the polynomial defining it. Let us denote this point by $p_Q$. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subspace

$$H_L \subset \mathbb{P}^9$$

such that

$L \subset Q$ iff and only if $p_Q \in H_L.$
(iii) Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that $L_1, L_2, L_3$ are contained in a quadric $Q$.

(iv) Explain (briefly) that if $L_1, L_2, L_3$ are disjoint lines, then $Q$ can be assumed to be irreducible.

Answer: 

(i) Consider a homogeneous degree 3 polynomial

$$F(X_0, X_1, X_2, X_3) = a_0X_0^2 + a_1X_0X_1 + \cdots + a_{10}X_3^2.$$ 

This has 4 square terms and \(\binom{4}{2} = 6\) mixed terms, i.e. 10 terms in total. We associate the quadric $Q := Z(F)$ the point

$$p_Q = [a_0 : a_1 : \ldots : a_{10}] \in \mathbb{P}^9.$$ 

Note that this association is independent of the defining equation of the quadric. Indeed, if $F$ and $G$ define the same quadric, then by the Nullstellensatz it follows that $G = cF$ for some constant $c \neq 0$. But then the point $p_Q$ does not change, since the coefficients $a_i$ are considered projectively. Finally, this association is clearly bijective, since for any $p_Q \in \mathbb{P}^9$ we can recover the equation of the quadric $Q$ (up to scalars).

(ii) Assume that the line $L$ is given by $X_0 = X_1 = 0$. Then

$$L \subset Q \cong F(0, 0, X_2, X_3) = 0.$$ 

Since

$$F(0, 0, X_2, X_3) = a_8X_2^2 + a_9X_2X_3 + a_{10}X_3^2,$$

we obtain

$$a_8 = a_9 = a_{10} = 0.$$ 

These equations define a codimension 3 projective space $H_L \subset \mathbb{P}^9$ such that

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$ 

If $L'$ is any line and $Q'$ is a quadric, we can change coordinates linearly so that $L'$ becomes the line $L: X_0 = X_1 = 0$. After the coordinate change, $Q'$ is mapped to another quadric $Q$ and the coefficients of $Q$ are linear combinations of coefficients of $Q'$. From above

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$
Note that
\[ L' \subset Q' \text{ if and only if } L \subset Q. \]
Therefore there exists a codimension 3 projective space \( H_{L'} \subset \mathbb{P}^9 \), the transformation of \( H_L \) via the change of coordinates, such that
\[ L' \subset Q' \text{ if and only if } p_{Q'} \in H_{L'}. \]

(iii) The codimension 3 subspaces \( H_L \) correspond to 7 dimensional linear subspaces \( A_L \subset \mathbb{A}^{10} \) cut out by the same 3 linear equations. Now,
\[ A_{L_1} \cap A_{L_2} \cap A_{L_3} \]
is described by 9 linear equations in \( \mathbb{A}^{10} \). Such a system of equations must have a nontrivial solution \( p \). This solution \( p \) considered projectively satisfies
\[ p \in H_{L_1} \cap H_{L_2} \cap H_{L_3}. \]
Now, \( p \) determines a quadric \( Q \) with \( p = p_Q \). It follows by (iii) that
\[ L_1 \cap L_2 \cap L_3 \subset Q. \]

(iv) If the quadric \( Q \) is reducible, then \( Q \) is the union of two hyperplanes \( H_1 \) and \( H_2 \). If \( L_1, L_2 \) and \( L_3 \) are 3 disjoint lines and
\[ L_1 \cup L_2 \cup L_3 \subset Q, \]
there should be two lines in the same hyperplane \( H_1 \cong \mathbb{P}^2 \) or \( H_2 \cong \mathbb{P}^2 \). But two lines on \( \mathbb{P}^2 \) always intersect in 1 point. This is a contradiction. Therefore we can assume \( Q \) is irreducible.

\[ \square \]

**Problem 6.** Let \( Q \) be a quadric in \( \mathbb{P}^2 \). Show that \( Q \cong \mathbb{P}^1 \) but \( S(Q) \not\cong S(\mathbb{P}^1) \).

**Answer:** After changing coordinates, we may assume that the quadric takes the form \( Q = \{ xy = z^2 \} \). This was seen to be isomorphic to \( \mathbb{P}^1 \) in class via the map \( [s : t] \rightarrow [s^2 : t^2 : st] \). The coordinate ring \( S(\mathbb{P}^1) = \mathbb{C}[s, t] \) and \( S(Q) = \mathbb{C}[x, y, z]/(xy - z^2) \).
Assuming that the two rings are isomorphic, consider the maximal ideal \( \mathfrak{m} = (x, y, z) \) in \( \mathbb{C}[x, y, z]/(xy - z^2) \). It corresponds to a maximal ideal in \( \mathbb{C}[s, t] \) so it must be of the form \( (s - a, t - b) \), hence generated by two elements. Thus \( \mathfrak{m} \) is generated by 2 elements \( f, g \) in \( \mathbb{C}[x, y, z]/(xy - z^2) \), which means
\[ (x, y, z) = (f, g, xy - z^2) \]
in \( \mathbb{C}[x, y, z] \). By looking at the part of degree 1 we obtain that \( \langle x, y, z \rangle \) is a vector space isomorphic to \( \langle f^{(1)}, g^{(1)} \rangle \). Comparing dimensions, we obtain a contradiction. \[ \square \]