Problem 1. (Intersections with hypersurfaces.)

(i) Show that if $X$ is a hypersurface in $\mathbb{P}^n$ and $Y \subset \mathbb{P}^n$ is a projective variety which is not a point, then the intersection of $X$ and $Y$ is non-empty.

(ii) If $X$ is a degree $d$ hypersurface in $\mathbb{P}^n$ and $L$ is a line in $\mathbb{P}^n$ not contained in $X$, show that $L$ and $X$ intersect in at most $d$ points. In fact, they intersect in exactly $d$ points counted with multiplicity.

Answer: (i) Suppose the intersection of $X$ and $Y$ is empty, then $Y \subset \mathbb{P}^n \setminus X$.

But $\mathbb{P}^n \setminus X$ is isomorphic to an affine in $\mathbb{A}^N$ for some $N$. In particular the coordinate functions $y_1, \ldots, y_N$ on $\mathbb{A}^N$ are regular on $\mathbb{P}^n \setminus X$. By contrast the regular functions on $Y$ are constant. Hence the restriction of $y_i$ to $Y$ are constant, hence $Y$ must be a point.

(ii) After a change of coordinates, we can assume that $L$ is cut out by the equations

$$X_2 = \ldots = X_n = 0.$$ 

Indeed, pick any two points $p$ and $q$ on $L$, change coordinates such that $p$ and $q$ become $[1 : 0 : \ldots : 0]$ and $[0 : 1 : \ldots : 0]$. It is clear that the line $L$ that $p$ and $q$ determine is given by $X_2 = \ldots = X_n = 0$.

Recall that

$$f = \sum_{i_0 + \ldots + i_n = d} a_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n}.$$ 

Let

$$F(X_0, X_1) = f(X_0, X_1, 0, \ldots, 0) = \sum_{i_0 + i_1 = d} a_{i_0 i_1 0 \ldots 0} X_0^{i_0} X_1^{i_1}.$$ 

Assuming

$$x = [x_0 : x_1 : \ldots : x_n] \in L \cap X$$

then $x_2 = \ldots = x_n = 0$ and $f(x) = 0$. Therefore

$$F(x) = \sum_{i_0 + i_1 = d} a_{i_0 i_1 0 \ldots 0} x_0^{i_0} x_1^{i_1} = 0.$$ 

Note that if $F \equiv 0$ it follows that $L \subset X$,

which is impossible. Therefore $F$ is a non-zero homogenous polynomial of degree at most $d$.

If $a_{d0 \ldots 0} = 0$, then $F = x_0 G$ where $G$ has degree $d - 1$. It follows by induction that $G$ has at most $d - 1$ roots and thus $F$ has at most $d$ roots, namely the zeros of $F$ and $[0 : 1 : \ldots : 0]$.

Assume now $a_{d0 \ldots 0} \neq 0$. If $x_1 = 0$, this implies $x_0 = 0$, which is not in $\mathbb{P}^n$. Therefore we can assume $x_1 = 1$ and $x_0$ is a solution of

$$\sum_{i_0 + i_1 = d} a_{i_0 i_1 0 \ldots 0} X_0^{i_0} = 0.$$ 

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This is a nonzero degree $d$ polynomial, with at most $d$ solutions. Therefore there are at most $d$ points in the intersection $L \cap X$.  

**Problem 2.** (Rational varieties.) The definition of birational isomorphisms given in class extends to the projective category. Two projective varieties $X$ and $Y$ are birational if there are rational maps

$$f : X \dasharrow Y, \quad g : Y \dasharrow X,$$

which are rational inverses to each other. Just as in the affine case, a birational isomorphism $f : X \dasharrow Y$ induces an isomorphism of the fields of rational functions $f^* : K(Y) \to K(X)$.

(i) Explain that if $X$ is rational, then

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X$ and $Y$ are rational, then $X \times Y$ is rational.

(iii) Show that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(iv) The group of automorphisms of the field of fractions in two variables $k(x, y)$ is called the Cremona group. Explain that the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$. Explain that the Cremona involution

$$(x, y) \to (x^{-1}, y^{-1})$$

extends to an automorphism of $k(x, y)$. What is the corresponding birational involution of $\mathbb{P}^2$? Where is this birational automorphism regular?

(v) More generally, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group.

**Answer:**

(i) Clearly, $\mathbb{A}^n$ is birational to $\mathbb{P}^n$, hence

$$K(\mathbb{P}^n) \cong K(\mathbb{A}^n) \cong k(t_1, \ldots, t_n).$$

Thus $X$ is rational iff

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Let $U \subset \mathbb{P}^n$ be the open set where the coordinate $x_0 \neq 0$. Similarly, let $V \subset \mathbb{P}^m$ be the open set where the coordinate $y_0 \neq 0$, and let $W$ be the open set in $\mathbb{P}^{n+m}$ where the first coordinate is non-zero. Define $\phi : U \times V \to \mathbb{P}^{n+m}$ as

$$[x_0 : x_1 : \ldots : x_n] \times [y_0 : y_1 : \ldots : y_m] \to \begin{bmatrix} 1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \ldots : \frac{x_n}{x_0} : \frac{y_1}{y_0} : \frac{y_2}{y_0}, \ldots : \frac{y_n}{y_0} \end{bmatrix}.$$

It is easy to check that $\phi$ establishes an isomorphism between $U \times V$ and $W$, with inverse

$$\psi : W \to U \times V, \quad [1 : z_1 : \ldots : z_{n+m}] \to [1 : z_1 : \ldots : z_n] \times [1 : z_{n+1} : \ldots : z_{n+m}].$$

Therefore $\phi$ and $\psi$ define birational isomorphisms between $\mathbb{P}^n \times \mathbb{P}^m$ and $\mathbb{P}^{n+m}$.

Finally, if $X$ and $Y$ are birational to $\mathbb{P}^n$ and $\mathbb{P}^m$, then $X \times Y$ is birational to $\mathbb{P}^n \times \mathbb{P}^m$ which in turn is birational to $\mathbb{P}^{n+m}$. Therefore, $X \times Y$ is rational.
(iii) Two closed subsets \( \{a\} \times \mathbb{P}^1 \) and \( \{b\} \times \mathbb{P}^1 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) have nonempty intersection. This is false in \( \mathbb{P}^2 \) by problem 1(iii). Hence \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) cannot be isomorphic.

(iv) We explained in (i) that
\[
K(\mathbb{P}^2) = k(x, y)
\]
hence any automorphism of \( k(x, y) \) corresponds to an automorphism of \( K(\mathbb{P}^2) \) which in turn gives a birational isomorphism of \( \mathbb{P}^2 \). The involution
\[
(x, y) \rightarrow (x^{-1}, y^{-1})
\]
corresponds to the birational map
\[
f[x : y : z] = [x^{-1} : y^{-1} : z^{-1}].
\]
This map is regular on \( \mathbb{P}^2 \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \). Indeed, to show that the map is regular at the points where \((x, y) \neq (0, 0)\), we rewrite it in the form
\[
f[x : y : z] = \begin{bmatrix} z & \frac{z}{x} & \frac{z}{y} & 1 \end{bmatrix}.
\]

(v) The automorphism
\[
(x, y) \rightarrow (x^a y^b, x^c y^d)
\]
where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})
\]
belongs to the Cremona group. Its inverse is
\[
(x, y) \rightarrow (x^{a'} y^{b'}, x^{c'} y^{d'})
\]
where
\[
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}
\]
is the inverse of the matrix above.

\(\square\)

**Problem 3.** *(Quadrics are rational.)*

\(i\) Show that a non-singular irreducible quadric \( Q \) in \( \mathbb{P}^3 \) can be written in the form
\[
xy = zw
\]
after a suitable change of homogeneous coordinates. Combining this result with the Segre embedding, conclude that any quadric in \( \mathbb{P}^3 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), hence it is rational.

\(ii\) In general, show that any non-singular irreducible quadric \( Q \subset \mathbb{P}^{n+1} \) is birational to \( \mathbb{P}^n \).
Answer: (i) A homogeneous quadratic polynomial \( f(x, y, z, w) \) can be written as

\[
f = \begin{bmatrix} x & y & z & w \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}
\]

where \( A \) is a symmetric 4 \( \times \) 4 matrix. Because \( A \) is symmetric, it can be diagonalized. Thus we can find one set of coordinates \( x_1, y_1, z_1, w_1 \) such that

\[
f = \begin{bmatrix} x_1 & y_1 & z_1 & w_1 \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 y_1^2 + \lambda_3 z_1^2 + \lambda_4 w_1^2.
\]

If two \( \lambda \)'s are zero then it is easy to see that the corresponding quadric is not irreducible. Assuming \( A \) is nondegenerate, then all \( \lambda \)'s are not zero. We can change coordinates again, setting

\[
x_2 = \sqrt{\lambda_1} x_1 + i \sqrt{\lambda_2} y_1, \quad y_2 = \sqrt{\lambda_1} x_1 - i \sqrt{\lambda_2} y_1, \\
z_2 = i(\sqrt{\lambda_3} z_1 + i \sqrt{\lambda_4} w_1), \quad w_2 = i(\sqrt{\lambda_3} z_1 - i \sqrt{\lambda_4} w_1).
\]

This is a change of coordinates since the matrix of coefficients

\[
\begin{pmatrix} \sqrt{\lambda_1} & i \sqrt{\lambda_2} & 0 & 0 \\ \sqrt{\lambda_1} & -i \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & i \sqrt{\lambda_3} & -\sqrt{\lambda_4} \\ 0 & 0 & i \sqrt{\lambda_3} & \sqrt{\lambda_4} \end{pmatrix}
\]

is invertible. Indeed, the determinant equals \( 4 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \neq 0 \). Then

\[
f = x_2 y_2 - z_2 w_2.
\]

We have seen in class that this quadric contains a one dimensional family of lines.

(ii) Similar to (i), we can assume that the quadric \( Q \) is defined by

\[
x_0 x_1 - x_2^2 - x_3^2 - \ldots - x_n^2 = 0.
\]

Pick the point \( p = [1 : 0 : \ldots : 0] \in Q \), and let \( H \) be the hyperplane \( X_0 = 0 \).

The line passing through \( p = [1 : 0 : \ldots : 0] \) and \( q = [x_0 : \ldots : x_n] \) is

\[
[r + sx_0 : sx_1 : \ldots : sx_n].
\]

This line intersects the hyperplane \( H \) when \( r + sx_0 = 0 \). Therefore the line intersects \( H \) at \( [0 : sx_1 : \ldots : sx_n] = [0 : x_1 : \ldots : x_n] \). This may be undefined when \( x_1 = x_2 = \cdots = x_n = 0 \), i.e. when \( q = p \). We obtain a morphism

\[
f : Q \setminus \{p\} \to H, \quad [x_0 : x_1 : \ldots : x_n] \to [0 : x_1 : \ldots : x_n].
\]

The rational inverse of \( f \) is given by

\[
g : H \longrightarrow Q, \quad [x_1 : x_2 : \ldots : x_n] = \left[ \frac{x_2^2 + \cdots + x_n^2}{x_1} : x_1 : x_2 : \ldots : x_n \right].
\]

This may be undefined at the points where \( x_1 = 0 \).

Since \( f \) has a rational inverse, \( Q \) is birational to \( H \).
Problem 4. (Intersection of quadrics.) Consider the following two quadrics in \( \mathbb{P}^3 \):

\[
Q_1 = \mathcal{Z}(xw - yz), \quad Q_2 = (yw - (x - z)(x - \lambda z)).
\]

Show that the intersection of the two quadrics is isomorphic to the elliptic curve \( \mathcal{E}_\lambda \subset \mathbb{P}^2 \).

**Answer:** Suppose \( p = [x : y : z : w] \in Q_1 \cap Q_2 \), then

\[
xw = yz, \quad yw = (x - z)(x - \lambda z).
\]

Multiplying the first equation by \( y \) and the second by \( x \) and substracting, we have

\[
y^2z = x(x - z)(x - \lambda z).
\]

We define

\[
f : Q_1 \cap Q_2 \to \mathcal{E}_\lambda, \quad [x : y : z : w] \to [x : y : z].
\]

Note that \( f \) is not well-defined at \( [0 : 0 : 0 : 1] \in Q_1 \cap Q_2 \). In this case, we set

\[
f([0 : 0 : 0 : 1]) = [0 : 0 : 1].
\]

We claim that \( f \) is bijective and its inverse is given by

\[
g : \mathcal{E}_\lambda \to Q_1 \cap Q_2, \quad [x : y : z] \to [x : y : z : \frac{y^2z}{x}].
\]

Note that \( g \) is undefined at \( [0 : 0 : 1] \) and \( [0 : 1 : 0] \). Since \( [0 : 1 : 0] = f([0 : 1 : 0 : 0]) \), and \( [0 : 0 : 1] = f([0 : 0 : 0 : 1]) \), we set

\[
g([0 : 1 : 0]) = [0 : 1 : 0 : 0], \quad g([0 : 0 : 1]) = [0 : 0 : 0 : 1].
\]

It is straightforward to check that \( f \) and \( g \) are inverses.

It is more delicate to check that \( f \) and \( g \) are actually morphisms. To prove this fact, you need to define

\[
f([x : y : z : w]) = \begin{cases} [x : y : z] & \text{if } (x, y, z) \neq (0, 0, 0) \\ [w(x - z)(x - \lambda z) : z(y - w)(y - \lambda w) : w^3] & \text{if } w \neq 0 \end{cases}.
\]

You will need to make sure that the function is defined everywhere on \( Q_1 \cap Q_2 \), and well-defined on overlaps. This last statement is equivalent to the equalities of rational functions

\[
\frac{w(x - z)(x - \lambda z)}{x} = \frac{z(y - w)(y - \lambda w)}{y} = \frac{w^3}{z}
\]

which can be derived from manipulating the equations of the quadric. The argument for the inverse is similar. We define

\[
g([x : y : z]) = \begin{cases} [x^2 : xy : xz : yz] & \text{when } [x : y : z] \neq [0 : 0 : 1] \text{ or } [0 : 1 : 0], \\ [xy : y^2 : yz : (x - z)(x - \lambda z)] & \text{when } [x : y : z] \neq [1 : 0 : 1] \text{ or } [1 : 0 : \lambda]. \end{cases}
\]

Since on \( \mathcal{E}_\lambda \):

\[
y^2z = x(x - z)(x - \lambda z),
\]

it follows that

\[
[x^2 : xy : xz : yz] = [xy : y^2 : yz : (x - z)(x - \lambda z)]
\]

on overlaps (indeed, the coordinates are proportional to each other with the proportionality factor \( y/x \)). Therefore, \( g \) is a well-defined morphism.

\[\square\]
Problem 5. (Twisted curves and complete intersections.)

A variety $Y$ of dimension $r$ in $\mathbb{P}^n$ is a strict complete intersection if $I(Y)$ can be generated by $n - r$ elements. $Y$ is a set-theoretic complete intersection if $Y$ can be written as the intersection of $n - r$ hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.

(ii) Show that the twisted cubic $T$ in $\mathbb{P}^3$ can be written as the set-theoretic intersection of the quadric and the cubic

$$Q = \mathcal{Z}(y^2 - xz), C = \mathcal{Z}(z^3 + xw^2 - 2yzw).$$

In particular, $T$ is a set theoretic complete intersection.

(iii) Show that the twisted cubic in $\mathbb{P}^3$ is the intersections of the three quadrics

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xt - yz), Q_3 = \mathcal{Z}(yt - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic.

(iv) Explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence $T$ is not a strict complete intersection.

Answer: (i) Let $f_1, \ldots, f_{n-r}$ be the generators of the ideal $I(Y)$ for a strict complete intersection $Y$. Then

$$Y = \mathcal{Z}(f_1) \cap \ldots \cap \mathcal{Z}(f_{n-r})$$

exhibits $Y$ as an intersection of $n - r$ hypersurfaces.

(ii) A point in the twisted cubic $T$ has coordinates $(t^3, t^2s, ts^2, s^3)$ so it clearly satisfies the equations of $Q$ and $C$. Conversely, pick a point in the intersection of $Q$ and $C$. We compute

$$(xw - yz)^2 = x^2w^2 + y^2z^2 - 2xyzw = x(xw^2 + z^3 - 2yzw) = 0 \implies xw = yz$$

$$(yw - z^2)^2 = y^2w^2 + z^4 - 2yzw^2 = zw^2 + z^4 - 2yzw^2 = z(xw^2 + z^3 - 2yzw) = 0 \implies yw = z^2.$$ 

By (iii), we know that such a point belongs to the twisted cubic.

(iii) The twisted cubic is given by

$$[x : y : z : t] = [a^3 : a^2b : ab^2 : b^3]$$

for some $[a : b] \in \mathbb{P}^1$. Clearly, points of these type satisfy the equations

$$xz = y^2, xt = yz, yt = z^2,$$

so the twisted cubic is contained in the intersection $Q_1 \cap Q_2 \cap Q_3$.

Conversely, given a point $p = [x : y : z : t]$ in the intersection of the three quadrics $Q_1, Q_2, Q_3$, set

$$[a : b] = [x : y]$$

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if $x \neq 0$. Otherwise, if $x = 0$, then $y = z = 0$ and set $[a : b] = [0 : 1]$. We claim
\[ [x : y : z : t] = [a^2 : a^2 b : a b^2 : b^3] \]
e.g. $p$ lies on the twisted cubic. This is clear for the case $x = 0$. If $x \neq 0$, then $a \neq 0$, and from the first and third equations we have
\[ y = x \cdot \frac{b}{a}. \]
Using the first and second equations, we solve
\[ z = x \cdot \frac{b^2}{a^2}, t = x \cdot \frac{b^3}{a^3}. \]
Then,
\[ [x : y : z : t] = \left[ x : \frac{b}{a} : x \frac{b^2}{a^2} : x \frac{b^3}{a^3} \right] = [a^2 : a b : a^2 b^2 : b^3]. \]
Finally, note that $[0 : 0 : 1 : 0] \in Q_1 \cap Q_2$ but not in $Q_3$; $[0 : 1 : 0 : 0] \in Q_2 \cap Q_3$ but not in $Q_1$; $[1 : 0 : 0 : 1] \in Q_3 \cap Q_2$ but not in $Q_1$. So any of the two quadrics can’t generate the twisted cubic.

(iv) It is clear that $I(T)$ cannot contain linear elements $ax + by + cz + dw$ because then $T$ would be contained in a hyperplane $ax + by + cz + dw = 0$ which would mean
\[ as^3 + bs^3 t + cst^2 + ds^3 = 0 \]
for all $s, t$ which clearly is impossible. Moreover, we have seen in (iii) that the ideal of $T$ contains 3 independent elements of degree 2. Since the space of homogeneous elements in $I(T)$ of degree 2 is 3 dimensional, any set of generators for $I(T)$ has to have at least 3 elements.

\[ \square \]

Problem 6. (Joins.) Let $G(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in the previous homework. Show that:

(i) The set \( \{(L, P) : P \in L\} \subset G(1, n) \times \mathbb{P}^n \) is closed.

(ii) If $Z \subset G(1, n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.

(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting $X$ and $Y$ is a closed subset of $\mathbb{P}^n$. It is called the join $J(X, Y)$ of $X$ and $Y$.

Answer: (i) We let
\[ J = \{(P, L) : P \in L\} \subset \mathbb{P}^n \times G(1, n). \]

We will think of lines $L$ in terms of their Plucker coordinates
\[ z_{ij} = a_i b_j - a_j b_i \]
where $a, b$ are two points on $L$ with
\[ a = [a_0 : \ldots : a_n], b = [b_0 : \ldots : b_n]. \]
In fact, it will be useful to form the vectors
\[ a = \sum a_i e_i, \quad b = \sum b_i e_i. \]

Similarly, a point \( P \in \mathbb{P}^n \) will have an associated vector
\[ p = \sum p_i e_i. \]

Now, if \( P \in L \), then \( p = sa + tb \) hence
\[ p \land a \land b = 0. \]

Then
\[
(\sum p_i e_i) \land (\sum a_i e_i) \land (\sum b_i e_i) = (\sum p_i e_i) \land \left( \sum z_{jk} e_j \land e_k \right)
\]
\[
= \sum_{i<j<k} (p_iz_{jk} - p_jz_{ik} + p_kz_{ij}) e_i \land e_j \land e_k.
\]

The conclusion is that \( J \) is defined by the equations
\[ p_iz_{jk} - p_jz_{ik} + p_kz_{ij} = 0 \]
which are bihomogeneous in the variables. Thus, \( J \) is projective.

(ii) Let
\[ p : J \to \mathbb{G}(1,n), \quad q : J \to \mathbb{P}^n \]
be the natural projections. Then, for any \( Z \) closed in \( \mathbb{G}(1,n) \), the preimage \( p^{-1}(Z) \) is also closed. Thus \( q(p^{-1}(Z)) \) is closed by the main theorem of projective varieties. This set consists in points \( P \) lying on lines \( L \) such that \( L \in Z \), hence it can be identified with the union of all lines in \( Z \).

(iii) We let \( A \) be the set of lines intersecting \( X \) and \( B \) be the set of lines intersecting \( Y \). We show \( A \) and \( B \) are closed in \( \mathbb{G}(1,n) \), hence so is \( Z = A \cap B \). The join \( J(X,Y) \) is simply the union of lines contained in \( Z \) hence it must be closed in \( \mathbb{P}^n \) by item (ii).

It suffices to prove \( A \) is closed in \( \mathbb{G}(1,n) \). Indeed, we can think of \( A \) as the projection under \( p \) of the set
\[ \{(P,L) : P \in L \} \cap X \times \mathbb{G}(1,n) = J \cap q^{-1}(X). \]

Hence
\[ A = p(J \cap q^{-1}(X)) \]
which is closed because \( p \) is closed and \( q \) is continuous. \( \square \)