
Problem 1. (Semicontinuity of fiber dimensions.) Assume that \( f : X \rightarrow Y \) is a surjective morphism of projective varieties. Show that \( Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} \) is closed.

Answer: Let \( n = \dim X, m = \dim Y \). We will use induction on \( m \), the case \( m = 0 \) being clear. Let \( d = n - m \geq 0 \). When \( k \leq d \) there is nothing to prove since \( Y_k = Y \) by the theorem of dimension of fibers. Asssume \( k > d \). Let \( U \) be an open subset over which \( \dim f^{-1}(y) = d \). The existence of \( U \) is proven in the theorem of dimension of fibers. Let \( Z = Y \setminus U \) is a closed subset of \( Y \), so \( \dim Z < \dim Z \Rightarrow \dim Z \leq m - 1 \). Note that \( Y_k \subset Z \) for \( k > d \). We wish to show that \( Y_k \) is closed in \( Z \), so then it will be closed in \( Y \) as well.

To this end, consider the restriction \( \bar{f} : W \rightarrow Z \), where \( W = f^{-1}(Z) \). We clearly have

\[
Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} = \{ y \in Z : \dim \bar{f}^{-1}(y) \geq k \}.
\]

We use induction on the dimension of the base to conclude. Indeed, \( \dim Z \leq m - 1 \), and working with each irreducible component of \( Z \) one at a time, we may assume \( Z \) is irreducible. In this case however the domain \( W = f^{-1}(Z) \) may have components \( W_1, \ldots, W_r \). Since \( Z \) is closed, \( W_i \) are closed in \( X \) hence projective in \( X \). Let \( \bar{f}_i : W_i \rightarrow Z \) be the restricted morphism. Then

\[
Y_k = \bigcup Y_k(\bar{f}_i)
\]

where the sets of the right are calculated with respect to the morphisms \( \bar{f}_i \). By the induction hypothesis applied to \( \bar{f}_i \), it follows that \( Y_k(\bar{f}_i) \) are closed in the image of \( \bar{f}_i \) which in turn is closed in \( Z \) by projective hypothesis, hence \( Y_k \) is closed in \( Z \) as well. \( \square \)

Problem 2. (Criterion for irreducibility.) Assume that \( f : X \rightarrow Y \) is a surjective morphism of projective algebraic sets such that \( Y \) is irreducible and all fibers of \( f \) are irreducible of the same dimension. Show that \( X \) is irreducible as well.

Answer: Write \( n \) for the common dimension of the fibers. Write \( X_1, \ldots, X_r \) for the irreducible components of \( X \). Then

\[
\bigcup f(X_i) = Y
\]

and \( f(X_i) \) are closed in \( Y \) since \( f \) is a morphism of projective sets, hence a closed map. Since \( Y \) is irreducible, there exists \( i \) such that \( f(X_i) = Y \). Assume that \( X_1, \ldots, X_s \) are chosen so that

\[
f(X_1) = \ldots = f(X_s) = Y
\]

but \( f(X_j) \neq Y \) for \( j > s \). Construct \( U_1, \ldots, U_s \) nonempty open sets in \( Y \) such that

\[
y \in U_i, 1 \leq i \leq s \Rightarrow \dim(f|_{X_i})^{-1}(y) = n_i = \dim X_i - \dim Y.
\]
In fact, even for \( j > s \) we can define \( U_j = Y \setminus f(X_j) \) and for \( y \in U_j \) we have
\[
(f|_{X_j})^{-1}(y) = \emptyset.
\]
Write
\[
U = \cap_{i=1}^r U_i,
\]
which is open and nonempty. For \( y \in U \), \( f^{-1}(y) \) is irreducible and nonempty, and is covered by \( X_1, \ldots, X_r \) so it must exist \( i_0 \) such that
\[
f^{-1}(y) \subset X_{i_0}.
\]
It is clear from the choice of \( i_0 \) that the entire fiber over \( y \) can be computed in \( X_{i_0} \) so that
\[
(f|_{X_{i_0}})^{-1}(y) = f^{-1}(y) \neq \emptyset
\]
so
\[
f|_{X_{i_0}} : X_{i_0} \to Y
\]
must be surjective by the definition of \( U_{i_0} \), and \( i_0 \leq s \). Furthermore \( n = n_{i_0} \) is the common dimension of the fibers since the fiber dimension can be calculated at \( y \) and
\[
(f|_{X_{i_0}})^{-1}(y) = f^{-1}(y).
\]
If \( z \in Y \), then
\[
(f|_{X_{i_0}})^{-1}(z) \subset f^{-1}(z)
\]
and the left hand side is at least of dimension \( n_{i_0} = \dim X_{i_0} - \dim Y \) by the theorem on dimension of fibers. But \( f^{-1}(z) \) is irreducible and \( n = n_{i_0} \) dimensional, so must have equality. Thus
\[
f^{-1}(z) \subset X_{i_0}
\]
for all \( z \in Y \). This shows that there are no components in \( X \) other than \( X_{i_0} \), so \( X \) is irreducible.

\[
\square
\]

**Problem 3.** *(Intersections in projective space.)* Let \( X \) and \( Y \) be two subvarieties of \( \mathbb{P}^n \). Show that if \( \dim X + \dim Y \geq n \), then \( X \cap Y \) is not empty.

**Answer:** Let \( H_1, H_2 \) be two disjoint linear subspaces of dimension \( n \) in \( \mathbb{P}^{2n+1} \). We write \([x_0 : x_1 : \ldots : x_n : y_0 : y_1 : \ldots : y_n]\) for the homogeneous coordinates in \( \mathbb{P}^{2n+1} \). Without loss of generality, we may assume \( H_1 \) is given by the equations
\[
y_0 = y_1 = \ldots = y_n = 0,
\]
while \( H_2 \) is given by
\[
x_0 = \ldots = x_{n+1} = 0.
\]
We regard
\[
X \subset H_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}, \quad Y \subset H_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}
\]
as subvarieties of \( \mathbb{P}^{2n+1} \). We form the join \( J(X,Y) \) in \( \mathbb{P}^{2n+1} \).
We first prove that $J(X,Y)$ has dimension $\dim X + \dim Y + 1$. Indeed, any point $P$ in $J(X,Y)$ lies on a line $L$ which intersects both $X$ and $Y$ in two points $Q$ and $R$. The map 

$$f : J(X,Y) \to X \times Y, P \mapsto (Q,R)$$

is a well-defined morphism since given $P$, then $Q$ and $R$ are uniquely defined. Indeed, if $P \in J(X,Y)$ has coordinates $[p_0 : \ldots : p_{2n+1}]$ then $Q = [p_0 : \ldots : p_n : 0 \ldots : p_n]$ and $R = [0 : \ldots : 0 : p_{n+1} : \ldots : p_{2n+1}]$, as claimed. The fibers of $f$ are lines $QR$, hence they are 1 dimensional. Thus 

$$\dim J(X,Y) = \dim(X \times Y) + 1 = \dim X + \dim Y + 1 \geq n + 1.$$ 

Even stronger, by the previous problem, $J(X,Y)$ is irreducible since it fibers over the irreducible set $X \times Y$ with equidimensional fibers.

Next, let $K_i$ be the hyperplane $x_i - y_i = 0$ for $0 \leq i \leq n$. We claim that 

$$X \cap Y \cong J(X,Y) \cap K_0 \cap K_1 \cap \ldots \cap K_n.$$ 

Indeed, any point $P \in J(X,Y)$ lies on a line $QR$ with $Q \in X, R \in Y$, hence 

$$P = \alpha Q + \beta R = [\alpha q : \beta r],$$

where $q$ and $r$ are the homogeneous coordinates of $Q$ and $R$ in $\mathbb{P}^n$. The requirement that 

$$P \in \bigcap_{0 \leq i \leq n} K_i$$

means that 

$$\alpha q_i = \beta r_i$$

hence $Q = R$. This means $P = Q = R \in X \cap Y$, proving the above equality.

Finally, Intersecting with a hyperplane either keeps the same dimension or cuts the dimension down by 1, hence 

$$\dim (J(X,Y) \cap K_0 \cap \ldots K_n) \geq 0 \implies X \cap Y \neq \emptyset.$$

□

Problem 4. (Lines on hypersurfaces.)

(i) Let $d > 2n - 3$. Show that a general degree $d$ hypersurface in $\mathbb{P}^n$ contains no lines.

(ii) Let $f$ be a degree 4 homogeneous polynomial in 4 variables and let $Z_f$ be the quartic surface $f = 0$ in $\mathbb{P}^3$. Show that there is a single polynomial $\Phi$ in the coefficients of $f$ which vanishes if and only if the quartic surface $Z_f \subset \mathbb{P}^3$ contains a line.
Answer: (i) We think of a hypersurface $X = Z(f)$ as a point in projective space $\mathbb{P}^N$ for $N = \binom{n+d}{d} - 1$, by means of the coefficients $a_I$ of its defining equation $f = \sum a_I X^I$.

We form the incidence correspondence $J = \{(L, X) : L \subset X\} \subset G(1, n) \times \mathbb{P}^N$ and we let $p : J \to G(1, n)$, $q : J \to \mathbb{P}^N$ be the two projections.

We claim that the fibers of $p$ have dimension $N - (d+1)$. Indeed, fix a line $L$ and study $p^{-1}(L)$. Without loss of generality, we may assume $L$ is given by the equations

$$x_0 = \ldots = x_{n-2} = 0.$$ 

If $X \in p^{-1}(L)$ is given by the polynomial $f = 0$, the requirement $L \subset X$ means $f(0 : \ldots : 0 : s : t) = 0$ for all $s, t$. In particular, the $d+1$ coefficients of $s^{i}t^{d-1}$ for $0 \leq i \leq d$ must vanish:

$$a_{0\ldots0,i,d-i} = 0,$$

while the other coefficients are arbitrary. Thus $p^{-1}(L)$ has codimension $d+1$ in $\mathbb{P}^N$, as claimed. Also the fibers of $p$ are irreducible so $J$ is irreducible as well by Problem 2.

With this understood, we conclude by looking at the fibers of $p$ that

$$\dim J = \dim G(1, n) + N - (d+1) = (2n - 2) + N - (d+1) < N.$$ 

Therefore, the morphism $q$ is not surjective. In particular, the image $q(J)$ is a proper subvariety of $\mathbb{P}^N$. For hypersurfaces $X$ belonging to the complement $\mathbb{P}^n \setminus q(J)$, the preimage $q^{-1}(X)$ is therefore empty, or in other words, for there are no lines lying on such hypersurfaces.

(ii) In this case, we have $d = 4, n = 3, N = 34$. Let $J = \{(L, X) : L \subset X\}$. In this case, the above computation show that $\dim J = N - 1$. We claim that the image $q(J)$ is a codimension 1 subvariety of $\mathbb{P}^N$. We complete the proof letting $\Phi$ be a polynomial cutting out $q(J)$.

To prove $q(J)$ is of dimension 33, assume otherwise, namely that the dimension is 32 or lower. By the theorem of dimension of fibers, for all $[X] \in q(J)$, the fiber
$q^{-1}([X])$ has dimension at least $33 - 32 = 1$. In other words all quartics that contain at least one line in fact contain infinitely many lines. One example is the quartic

$$x^4 + y^4 + z^4 + w^4 = 0.$$ 

By symmetry, we may search for lines of the form $x = az + bw, y = cz + dw$ and substituting we find

$$(az + bw)^4 + (cz + dw)^4 + z^4 + w^4 = 0.$$ 

This gives

$$a^4 + c^4 + 1 = b^4 + d^4 + 1 = 0, a^2b^2 + c^2d^2 = 0, a^3b + c^3d = 0, ab^3 + cd^3 = 0.$$ 

We claim that there are finitely many solutions for $a, b, c, d$. If $a = 0$ then it is easy to conclude that $d = 0$ and $b, c$ have to satisfy $b^4 = c^4 = -1$, and the solution set is finite. Assume now that neither $a, b, c, d$ is zero. Then

$$a^3b = -c^3d, \quad ab^3 = -cd^3 \implies (a/b)^2 = (c/d)^2$$

and in addition $(ab)^2 = -(cd)^2$ so multiplying we find $a^4 = -c^4$ which contradicts $a^4 + c^4 = -1$. 

$\Box$