Problem 1. Show that the Segre embedding
\[ \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} \]
has degree \(\binom{n+m}{n}\).

Answer: Let \(\Sigma_{m,n}\) be the image of the Segre embedding. Degree \(\ell\) homogeneous polynomials on \(\mathbb{P}^{(n+1)(m+1)-1}\) restrict to \(\Sigma_{m,n}\) as polynomials in the variables \(x_i\) on \(\mathbb{P}^n\) and \(y_j\) on \(\mathbb{P}^m\), bihomogeneous of degree \(\ell\). The dimension of \(S(\Sigma_{m,n})^{(\ell)}\) then equals
\[ \binom{n+m}{n} \binom{n+m}{m}. \]
Expanding, we have
\[ \binom{n+m}{n} \binom{n+m}{m} = \frac{1}{n!m!} \cdot \ell^{n+m} + \text{l.o.t.} \]
which shows that the degree of \(\Sigma_{m,n}\) equals
\[ \frac{(n+m)!}{n!m!} = \binom{n+m}{n}. \]
\(\square\)

Problem 2. Let \(X \subset \mathbb{P}^n\) be a projective scheme with Hilbert polynomial \(\chi_X\). Define the arithmetic genus of \(X\) to be
\[ p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1). \]

(i) Show that the genus of \(\mathbb{P}^n\) is zero.

(ii) If \(X\) is a hypersurface of degree \(d\) in \(\mathbb{P}^n\), show that \(p_a(X) = \binom{d-1}{n}\). In particular, a cubic in \(\mathbb{P}^2\) has genus 1.

(iii) If \(X\) is a complete intersection of two surfaces of degree \(a\) and \(b\) in \(\mathbb{P}^3\) then
\[ p_a(X) = \frac{1}{2}ab(a+b-4) + 1. \]

In particular, intersection of two quadrics in \(\mathbb{P}^3\) has genus 1.

Answer: (i) We calculated the Hilbert polynomial of \(\mathbb{P}^n\) to be \(\chi(\ell) = \binom{\ell+n}{n}\). This yields immediately that \(p_a(\mathbb{P}^n) = 0\).

(ii) We calculated the Hilbert polynomial of a degree \(d\) hypersurface to be
\[ \chi(\ell) = \binom{n+\ell}{n} - \binom{n+\ell-d}{n}. \]
This yields
\[ p_a(X) = (-1)^{n-1}\left(1 - \binom{n-d}{n} - 1\right) = (-1)^n \binom{n-d}{n} = \binom{d-1}{n}. \]
(iii) We claim that the Hilbert polynomial of the complete intersection equals
\[
\chi(\ell) = \binom{\ell + 3}{3} - \binom{\ell + 3 - a}{3} - \binom{\ell + 3 - b}{3} + \binom{\ell + 3 - a - b}{3}.
\]
Then, we find
\[
\chi(0) = 1 - \binom{3 - a}{a} - \binom{3 - b}{b} + \binom{3 - a - b}{3}
\]
which yields the answer.

The claim about the Hilbert polynomial is justified as follows. Let \( f \) and \( g \) be the equations of the two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) whose intersection is \( X \). There is an exact sequence
\[
0 \to S(\mathbb{P}^3)^{(\ell - a - b)} \to S(\mathbb{P}^3)^{(\ell - a)} \oplus S(\mathbb{P}^3)^{(\ell - b)} \to S(\mathbb{P}^3)^{(\ell)} \to S(X)^{(\ell)} \to 0
\]
where the first two maps are given by
\[
P \mapsto (gP, fP)
\]
and
\[
(P, Q) \mapsto fP - gQ
\]
and the last map is the restriction. We conclude by considering dimensions.

\[\square\]

**Problem 3.** Given four general lines in \( \mathbb{P}^3 \), show that there are exactly 2 lines which intersect all four of them.

**Answer:** Recall that the space of lines in \( \mathbb{P}^3 \) is parametrized by the Grassmannian \( G = G(1, 3) \) which can be realized as a quadric in \( \mathbb{P}^5 \) via the Plucker embedding. For each line \( L_i \) define
\[
X_i = \{M \text{ line in } \mathbb{P}^3 : M \cap L_i \neq \emptyset\} \subset G(1, 3) \subset \mathbb{P}^5.
\]
We claim that
\[
X_i = H_i \cap G
\]
for a hyperplane \( H_i \) in \( \mathbb{P}^5 \). Indeed, working in Plucker coordinates, assume that \( L = L_i \) has coordinates \( l_{ij} \) and \( M \) has Plucker coordinates \( m_{kl} \). If these are calculated with respect to points \( A, B \) over \( L \) and points \( C, D \) on \( M \) then we have
\[
a \wedge b = \sum_{ij} l_{ij} e_i \wedge e_j
\]
\[
c \wedge d = \sum_{kl} m_{kl} e_k \wedge e_l.
\]
The requirement that \( L \) and \( M \) meet is equivalent to
\[
a \wedge b \wedge c \wedge d = 0
\]
since the vector space spanned by $a, b, c, d$ is 3 dimensional in this case. This gives

$$
\left( \sum_{ij} l_{ij} e_i \wedge e_j \right) \wedge \left( \sum_{kl} m_{kl} e_k \wedge e_l \right) = 0
$$

which gives

$$
l_{12}m_{34} - l_{13}m_{24} + l_{14}m_{13} - l_{24}m_{34} + l_{34}m_{12} = 0.
$$

This is clearly a linear equation in the variables $m_{kl}$ for each fixed $l_{ij}$.

Now, the lines $M$ that intersect $L_1, L_2, L_3, L_4$ are found as the intersection points

$$
X_1 \cap X_2 \cap X_3 \cap X_4 \subset G(1, 3).
$$

In other words, these points correspond to

$$
H_1 \cap H_2 \cap H_3 \cap H_4 \cap G(1, 3) \subset \mathbb{P}^5.
$$

We claim that this intersection consists of 2 points in general.

We claim first that the intersection $H_1 \cap H_2 \cap H_3 \cap H_4$ is a line $\ell$ in $\mathbb{P}^5$ in general. In any case, the intersection is given as the null space of the $4 \times 6$ matrix of coefficients describing the hyperplanes $H_i$. In general, this null space is 1 dimensional, so the intersection is a line, but it can also be that the null space has dimension 2 or higher. This condition is described as the rank of the matrix being 4 or less – in turn this is given by the vanishing of the $4 \times 4$ minors, so it is a closed subset $Z$ in the space $G \times G \times G \times G$. We assume $(L_1, L_2, L_3, L_4)$ are chosen away from $Z$.

Next, if $\ell$ is the intersection line, we claim it intersects the quadric $G$ in $\mathbb{P}^5$ in 2 points. Indeed, we may assume that after a change of coordinates, this line is given by $x_2 = x_3 = x_4 = x_5 = 0$. The quadric $G$ will be given by $\sum a_{ij} x_i x_j$ and the intersection of the line $\ell$ is obtained by solving

$$
a_{00}x_0^2 + a_{11}x_1^2 + a_{01}x_0x_1 = 0
$$

which has exactly two solutions. The only exceptions correspond to

$$
a_{01}^2 = 4a_{00}a_{11}
$$

which corresponds to one solution, or the case $a_{00} = a_{01} = a_{11} = 0$ which corresponds to infinitely many solutions. These are closed conditions determining a closed set $W$ as one can check.

Setting $U = G \setminus (Z \cup W)$ we obtain that for $(L_1, L_2, L_3, L_4)$ in $U$, there are exactly 2 lines intersecting $L_i$. Then $U$ is dense in $G \times G \times G \times G$ if nonempty. To show nonemptiness, we can pick 4 lines

$$
L_1 = \{x_0 = x_1 = 0\}, L_2 = \{x_0 = x_2 = 0\}, L_3 = \{x_0 + x_1 = x_2 + x_3 = 0\}, L_4 = \{x_0 + 2x_1 = x_2 + 2x_3 = 0\}.
$$
We claim this quadruple lies in $U$. Indeed, one can easily run the argument above to find the equations of the hyperplanes $H_1, H_2, H_3, H_4$ above in terms of the Plucker coordinates. We obtain

$$m_{01} = 0, m_{02} = 0, m_{13} + m_{03} + m_{12} + m_{02} = 0, 4m_{13} + 2m_{03} + 2m_{12} + m_{02} = 0.$$ 

We also have

$$m_{01}m_{23} - m_{02}m_{13} + m_{03}m_{12} = 0$$

for the equation of the quadric. These equations only have 2 common solutions as one checks immediately. 

**Problem 4.** Let $X$ be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree $d$ and codimension $c$ in $\mathbb{P}^n$.

(i) (Intersecting $X$ with hyperplanes to cut down the dimension), show inductively that

$$d \geq c + 1.$$ 

(iii) Show that equality holds for rational normal curves in $\mathbb{P}^n$, and for the image $v(\mathbb{P}^2)$ of the Veronese embedding

$$v : \mathbb{P}^2 \to \mathbb{P}^5.$$ 

(iii) Can you classify the varieties of degree 2?

**Answer:**

(i) Consider a hyperplane $H$ and consider the scheme $X \cap H$. By Bezout this has the same degree $d$ as $X$, and it has codimension $c$ in $H$, hence the inequality to prove for $X$ is equivalent to the inequality to prove for $X \cap H$ (which is still nondegenerate). By induction, we reduce to the case when $X$ consists of $d$ points in $\mathbb{P}^n$. In this case, we need to show $d \geq n + 1$ which is clear since if $d \leq n$, then $X$ would be degenerate, as any $n$ points are contained in a hyperplane $H$. This last statement can be seen by arranging that the $n$ points be $p_i = [0 : \ldots : 1 : 0 : \ldots : 0]$ for $0 \leq i \leq n - 1$, after a linear change of coordinates, and setting $H = \{x_n = 0\}$.

(ii) Both cases are particular examples of the Veronese embedding whose degree we calculate below.

Consider Veronese embedding

$$v_d : \mathbb{P}^n \to \mathbb{P}^N$$

constructed from degree $d$ monomials. We claim that the image $V_d$ has degree $d^n$. Indeed, degree $\ell$ polynomials in $N + 1$ variables become, after restricting to $V_d$, polynomials of degree $d\ell$ on $\mathbb{P}^n$. Hence the Hilbert function of $V_d$ equals

$$\chi(\ell) = \binom{d\ell + n}{n} = d^n \frac{\ell^n}{n!} + \text{l.o.t.}$$
confirming the claim.

Now, it is easy to see that

\[ d^n = \text{codim } V_d + 1 = \binom{d+n}{n} - n \]

holds for \( n = 1 \) or for \( d = n = 2 \).

(iii) After passing to a smaller projective space, we may assume that \( X \) is nondegenerate. Degree \( d = 2 \) forces \( c = 1 \) hence \( X \) is isomorphic to a projective quadric.