Math 203, Problem Set 2. Due Friday, October 12.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

1. An algebraic set \( Z \subset \mathbb{A}^2 \) defined by an irreducible polynomial \( f \) of degree 2 is called an irreducible conic.

   (i) Show that any irreducible conic can be written in the form
   \[
   Y - X^2 = 0 \text{ or } XY - 1 = 0
   \]
   after an affine change of coordinates in \( \mathbb{A}^2 \).

   \textbf{Remark:} An affine change of coordinates taking \((x, y)\) into \((X, Y)\) is a transformation of the form
   \[
   \begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + b,
   \]
   where \( A \) is a \( 2 \times 2 \) invertible matrix and \( b \in \mathbb{A}^2 \) is a vector.

   \textbf{Hint:} Write \( f(x, y) = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6 \), and complete the square.

   (ii) Let \( Z_1 \) and \( Z_2 \) be two distinct irreducible conics in \( \mathbb{A}^2 \). Using (i), show that \( Z_1 \) and \( Z_2 \) intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

2. Find the coordinate rings of the following affine algebraic sets and decide which of the following algebraic sets are isomorphic, and which ones are not:

   (i) \( \mathbb{A}^1 \)
   (ii) \( Z(xy) \subset \mathbb{A}^2 \)
   (iii) \( Z(x^2 + y^2) \subset \mathbb{A}^2 \)
   (iv) \( Z(x^2 - y^5) \subset \mathbb{A}^2 \)
   (v) \( Z(y - x^2, z - x^3) \subset \mathbb{A}^3 \).

3. (\textit{Hartogs theorem and quasi-affine algebraic sets.}) Show that the quasi-affine set \( X = \mathbb{A}^2 \setminus \{(0, 0)\} \) is not isomorphic to an affine algebraic set.

   \textbf{Hint:} Argue by contradiction. Using your knowledge about the regular functions on \( X \), what can you say about the inclusion \( \iota : X \to \mathbb{A}^2 \)?

4. Let \( n \geq 2 \), and let \( S = \{a_1, \ldots, a_n\} \) be a finite set with \( n \) elements in \( \mathbb{A}^1 \).
(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take $X$ to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$ 

(ii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

*Hint:* Assume that $\Phi : A(X) \rightarrow k[t, t^{-1}]$ is an isomorphism. Observe that $X_i$ are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers $m_i$. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of $i$.

5. Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathbb{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathbb{Z}(y^n - x^n(x + 1)).$$

Show that $Z_n$ and $W_n$ are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \rightarrow Z_n, \quad f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \rightarrow Z_n \setminus \{(0, 0)\}.$$ 

Similarly, show that

$$g : \mathbb{A}^1 \rightarrow W_n, \quad g(t) = (t^n - 1, t^{n+1} - t)$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and $W_n$ where $g$ becomes an isomorphism.

(ii) Using (i), explain why $Z_n$ and $W_n$ are birational.

(iii) Assume that there exists an isomorphism

$$h : Z_n \rightarrow W_n$$

such that $h((0, 0)) = (0, 0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0, 0)\} \rightarrow W_n \setminus \{(0, 0)\}.$$ 

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$. 

(iv) *(Optional.*) Repeat the argument above without the assumption that $h$ sends the origin to itself. You may need to prove a stronger version of Problem 4.

6. *(Quotients.*) Taking quotients in algebraic geometry is subtle. We will explain how to take quotients by finite groups.

Let $X$ be an affine variety, and let $G$ be a finite group. Assume that $G$ acts on $X$ algebraically, i.e. that for every $g \in G$, we are given a morphism $g : X \to X$ (denoted by the same letter for simplicity of notation), such that

$$(gh)(p) = g(h(p))$$

for all $g, h \in G$ and $p \in X$.

(i) Let $g \in G$ act on the coordinate rings $A(X)$ via

$$f \mapsto f^g$$

with $f^g(p) = f(g(p))$.

Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all $G$-invariant functions on $X$. Show that $A(X)^G$ is a finitely generated $k$-algebra.

(ii) By (i), there is an affine variety $Y$ with coordinate ring $A(X)^G$, together with a morphism

$$\pi : X \to Y$$

determined by the inclusion

$$A(X)^G \hookrightarrow A(X).$$

Show that $Y$ can be considered as the quotient of $X$ by $G$, denoted $X/G$, in the following sense: if $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(iii) Let

$$\mu_n = \left\{ \exp \left( \frac{2\pi i k}{n} \right), k \in \mathbb{Z} \right\}$$

be the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Describe $\mathbb{C}/\mu_n$ and $\mathbb{C}^2/\mu_n$ as affine algebraic sets.