For this problem set, you may assume that the ground field is algebraically closed.

1. (Intersections of affine opens.) If $X$ is a variety, and $U, V$ are two affine open sets, then $U \cap V$ is an affine variety.

   *Hint:* Construct $U \cap V$ as $(U \times V) \cap \Delta$ where $\Delta \subset X \times X$ is the diagonal.

2. (Cubic curves are not rational.) We claimed in a previous lecture, but did not prove, that (most) cubic plane curves are not rational.

   Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation
   
   $$y^2 - x(x - 1)(x - \lambda) = 0.$$ 

   Show that $E_\lambda$ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps $F : \mathbb{A}^1 \rightarrow E_\lambda$.

   (i) Write
   
   $$F(t) = \left( \frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right)$$
   
   where the pairs of polynomials $(f, g)$ and $(p, q)$ have no common factors. Conclude that
   
   $$\frac{p^2}{q^2} = \frac{f(f - g)(f - \lambda g)}{g^3}$$
   
   is an equality of fractions that cannot be further simplified. Conclude that $f, g, f - g, g - \lambda g$ must be perfect squares.

   (ii) Conclude by proving the following:

   *Lemma:* If $f, g$ are polynomials in $k[t]$ without common factors and such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f - g, f - \lambda g$ are perfect squares, then $f$ and $g$ must be constant.

   *Hint:* Descent. Write $f = u^2, g = v^2$. Considering $f - g$ and $f - \lambda g$, prove that $u - v, u + v, u - \mu v, u + \mu v$ are also squares for some constant $\mu \neq 0, 1$. Show that suitable $\tilde{u}, \tilde{v}$ obtained as a linear combination of $u$ and $v$ verify the lemma, yet they have smaller degree than $\max(\deg f, \deg g)$.

   (iii) *Summary:* In $\mathbb{P}^2$, explain briefly that lines and irreducible conics are isomorphic to $\mathbb{P}^1$, while the elliptic (cubic) curve $E_\lambda \subset \mathbb{P}^2$:

   $$y^2 z = x(x - z)(x - \lambda z), \ \lambda \neq 0, 1$$

   is not.
3. (Isomorphisms of the projective line.)

(i) Show that every isomorphism \( f : \mathbb{A}^1 \to \mathbb{A}^1 \) is of the form \( f(x) = ax + b \).

(ii) Show that every isomorphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is of the form \( f(x) = \frac{ax + b}{cx + d} \) for some \( a, b, c, d \in k \), where \( x \) is an affine coordinate on \( \mathbb{A}^1 \subset \mathbb{P}^1 \).

(iii) Given three distinct points \( P_1, P_2, P_3 \in \mathbb{P}^1 \) and three distinct points \( Q_1, Q_2, Q_3 \in \mathbb{P}^1 \), show that there is a unique isomorphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( f(P_i) = Q_i \) for \( i = 1, 2, 3 \).

4. (Conics through 5 points.) In class, we have used conics as examples of projective varieties.

(i) Extend the result of 3(iii) to \( \mathbb{P}^2 \) as follows. Four points in \( \mathbb{P}^2 \) are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if \( p_1, \ldots, p_4 \) are points in general position, there exists a linear change of coordinates \( T : \mathbb{P}^2 \to \mathbb{P}^2 \) with \( T([1 : 0 : 0]) = p_1, T([0 : 1 : 0]) = p_2, T([0 : 0 : 1]) = p_3, T([1 : 1 : 1]) = p_4 \).

(ii) Given five distinct points in \( \mathbb{P}^2 \), no three of which are collinear, show that there is an unique irreducible projective conic passing through all five points. You may want to use part (i) to assume that four of the points are \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1] \).

Deduce that two distinct irreducible conics in \( \mathbb{P}^2 \) cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

Remark: A degree \( d \) rational curve in \( \mathbb{P}^2 \) is a the image of a morphism \( f : \mathbb{P}^1 \to \mathbb{P}^2 \) given by degree \( d \) polynomials. Lines have degree 1, conics have degree 2, etc.

For any degree \( d \), fix \( 3d - 1 \) points in \( \mathbb{P}^2 \) in “general position”. You may ask how many rational curves of degree \( d \) in \( \mathbb{P}^2 \) pass through these \( 3d - 1 \) points. Clearly, there is \( N_1 = 1 \) line through 2 points, and we have shown that \( N_2 = 1 \) conic through 5 points. The next few numbers are

\[ N_3 = 12, N_4 = 620, N_5 = 87, 304, N_6 = 26, 312, 976, N_7 = 14, 616, 808, 192. \]

Thus, there are are 12 rational cubics through 8 points, 620 rational quartics through 11 points and so on. A general answer for arbitrary \( d \) was found in 1994 using ideas from physics/string theory. The area of algebraic geometry that computes these numbers is called enumerative geometry/Gromov-Witten theory.

5. (Homogeneous coordinate rings.) Show that if \( Q \) is an irreducible conic in \( \mathbb{P}^2 \), then \( Q \cong \mathbb{P}^1 \) but the homogeneous coordinate rings \( S(Q) \not\cong S(\mathbb{P}^1) \). Thus the homogeneous coordinate ring is not invariant under isomorphism.
6. (*Twisted curves and complete intersections.*)

A variety $Y$ of dimension $r$ in $\mathbb{P}^n$ is a *strict complete intersection* if the ideal $I(Y)$ can be generated by $n - r$ homogeneous elements. $Y$ is a *set-theoretic complete intersection* if $Y$ can be written as the intersection of $n - r$ hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.

(ii) Show that the rational normal curve of degree three $T \subset \mathbb{P}^3$ (called the twisted cubic) can be written as the set-theoretic intersection of the quadric and the cubic

$$Q = Z(y^2 - xz), \ C = Z(z^3 + xw^2 - 2yzw).$$

In particular, $T$ is a set theoretic complete intersection.

(iii) Show that $T$ is the intersections of the three quadrics

$$Q_1 = Z(xz - y^2), \ Q_2 = Z(xw - yz), \ Q_3 = Z(yw - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic. In particular $I(T)$ contains three elements of degree 2.

(iv) Possibly using (iii), explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence $T$ is not a strict complete intersection.

*Remark:* It is an unsolved problem to show that every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic complete intersection. This is called the *Hartshorne conjecture.*

7. (*Introduction to moduli theory.*) Show that for any 3 lines $L_1, L_2, L_3$ in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric $Q$ and not on the polynomial defining it. Let us denote this point by $p_Q$. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

*Remark:* The projective space $\mathbb{P}^9$ is said to be the moduli space of quadrics in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subspace

$$H_L \subset \mathbb{P}^9$$

such that

$$L \subset Q \text{ iff and only if } p_Q \in H_L.$$  

*Hint:* You may want to change coordinates to assume that the line $L$ is cut out by the equations $X_0 = X_1 = 0$. This will force 3 of the coefficients of $Q$ to be zero. Which ones? What is the subspace $H_L$ in this case?
(iii) Show that any three codimension 3 projective linear subspaces of \( \mathbb{P}^9 \) intersect. In particular, show that
\[
H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,
\]
and conclude that \( L_1, L_2, L_3 \) are contained in a quadric \( Q \).