Math 203A - Solution Set 3

Problem 1. If $X$ is a variety, and $U,V$ are two affine open sets, then $U \cap V$ is an affine variety.

Answer: Let $\Delta \subset X \times X$ be the diagonal. Let $p_1, p_2 : X \times X \to X$ be the two projections. Clearly, $U \times V = p_1^{-1}(U) \times p_2^{-1}(V)$ is open in $X \times X$. Thus

$$U \cap V = (U \times V) \cap \Delta$$

is closed in $U \times V$. By the construction of products given in class, $U \times V$ is affine since $U,V$ are. It follows that $U \cap V$ is also affine being closed in an affine set.

We show that $U \cap V$ is irreducible. Indeed, $U, V$ are dense in $X$. The same must be true about $U \cap V$. Indeed, let $W$ be any open set. Then since $V$ is dense, $V \cap W$ is open and nonempty. Since $U$ is dense it intersects $V \cap W$ so $U \cap V \cap W \neq \emptyset$. Thus $U \cap V$ intersects any nonempty open set $W$. Thus

$$U \cap V = X,$$

which by assumption is irreducible. By Problem 7 in PSet 1, it follows $U \cap V$ is irreducible.

Problem 2. Let $\lambda \in k \setminus \{0,1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x - 1)(x - \lambda) = 0.$$

Show that $E_\lambda$ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \longrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)}\right)$$

where the pairs of polynomials $(f,g)$ and $(p,q)$ have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f - g)(f - \lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. Conclude that $f, g, f - g, g - \lambda g$ must be perfect squares.

(ii) Prove the following:

Lemma: If $f,g$ are polynomials in $k[t]$ without common factors and such that there is a constant $\lambda \neq 0,1$ for which $f,g, f - g, f - \lambda g$ are perfect squares, then $f$ and $g$ must be constant.
(iii) **Summary:** In $\mathbb{P}^2$, explain briefly that lines and irreducible conics are isomorphic to $\mathbb{P}^1$, while the elliptic (cubic) curve $E_\lambda \subset \mathbb{P}^2$:

$$y^2z = x(x-z)(x-\lambda z), \; \lambda \neq 0, 1$$

is not.

**Answer:**

(i) We have

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}.$$ 

Since $p, q$ are relatively prime, the right hand side cannot be further simplified. Similarly, $f, g, f-g, f-\lambda g$ cannot have any common factors. Indeed, a common factor for instance of $f$ and $f-g$, will necessarily have to divide $f - (f-g) = g$ as well. But this is impossible since $f$ and $g$ are coprime. Therefore the right hand side cannot be further simplified as well. Thus, for some constant $a \in k$, we must have

$$ap^2 = f(f-g)(f-\lambda g), \; aq^2 = g^3.$$ 

The exponents of the irreducible factors of the left hand sides of the above equations must be even. Therefore, the same must be true about the right hand side. This immediately implies that $g$ is a square. But since $f, f-g, f-\lambda g$ have no common factors, it follows that the exponents of the irreducible factors of each $f, f-g$ and $f-\lambda g$ must be even as well. Thus $f, f-g, f-\lambda g$ must be squares.

(ii) Pick $f$ and $g$ such that $\max(\deg f, \deg g)$ is minimal among all pairs $(f,g)$ which satisfy the requirement that $f, g, f-g, f-\lambda g$ are squares for some $\lambda \neq 0, 1$. We may assume that $f, g$ are coprime since otherwise we can reduce their degree by dividing by their gcd which is also a square. Write

$$f = u^2, \; g = v^2,$$

where $u, v$ are coprime. Then

$$f - g = (u-v)(u+v)$$

is a square. Note that $u-v$ and $u+v$ cannot have common factors since such factors will have to divide both

$$\frac{1}{2}(u-v) + (u+v) = u \text{ and } \frac{1}{2}((u-v) - (u+v)) = v$$

which is assumed to be false. Thus $u-v$ and $u+v$ are coprime, and since their product $f-g$ is a square, it follows that $u-v, u+v$ must be square. The same argument applied to

$$f - \lambda g = (u - \sqrt{\lambda}v)(u + \sqrt{\lambda}v)$$
shows that $u - \sqrt{\lambda}v, u + \sqrt{\lambda}v$ are squares. Let
\[ \tilde{u} = \frac{1 + \sqrt{\lambda}}{2} (u + v), \tilde{v} = \frac{\sqrt{\lambda} - 1}{2} (u - v). \]
Clearly, $\tilde{u}, \tilde{v}$ are squares. A direct computation shows that
\[ \tilde{u} - \tilde{v} = u + \sqrt{\lambda}v, \]
which is also a square. Finally,
\[ \tilde{u} - \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right)^2 \tilde{v} = \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} (u - \sqrt{\lambda}v) \]
is a square. Setting
\[ \mu = \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right)^2, \]
we see that $\tilde{u}, \tilde{v}, \tilde{u} - \tilde{v}, \tilde{u} - \mu \tilde{v}$ are squares. Note that $\mu \neq 0,1$ for $\lambda \neq 0,1$.
Furthermore,
\[ \max(\deg \tilde{u}, \deg \tilde{v}) = \frac{1}{2} \max(\deg f, \deg g). \]
Unless $f$ and $g$ are irreducible, this contradicts the assumption that $f, g$ are of minimal degree.

(iii) Lines: let $L$ be the line $\alpha x + \beta y + \gamma z = 0$. We may assume that $\gamma \neq 0$, eventually relabeling the coordinates if necessary. Set
\[ f : \mathbb{P}^1 \to L, [x : y] \mapsto \left[ x : y : -\frac{\alpha}{\gamma} x - \frac{\beta}{\gamma} y \right], \]
and
\[ g : L \to \mathbb{P}^1 [x : y : z] \to [x : y]. \]
It is easy to see that both $f$ and $g$ are well defined inverse isomorphisms.

Conics: changing coordinates, we may assume the conic $C$ is given by the equation
\[ xz = y^2. \]
We saw in class this is isomorphic to $\mathbb{P}^1$.

Cubics: assume that
\[ F : \mathbb{P}^1 \to \mathbb{E}_\lambda \subset \mathbb{P}^2 \]
is a morphism. Let $E_\lambda$ be the affine piece where $z = 1$. Note that $\mathbb{E}_\lambda \setminus E_\lambda = \{[0 : 1 : 0]\}$. The preimage $F^{-1}([0 : 1 : 0])$ is a closed subset of $\mathbb{P}^1$ so it is a finite set $S$. Restricting to $E_\lambda$, we obtain a morphism $\tilde{F} : \mathbb{P}^1 \setminus S \to E_\lambda$. In other words, there is a rational map $f : \mathbb{A}^1 \dashrightarrow E_\lambda$ which must be constant. This implies $F$ is constant.
Problem 3.  
(i) Show that every isomorphism \( f : \mathbb{A}^1 \to \mathbb{A}^1 \) is of the form \( f(x) = ax + b \).
(ii) Show that every isomorphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is of the form \( f(x) = \frac{ax+b}{cx+d} \) for some \( a, b, c, d \in k \), where \( x \) is an affine coordinate on \( \mathbb{A}^1 \subset \mathbb{P}^1 \).
(iii) Given three distinct points \( P_1, P_2, P_3 \in \mathbb{P}^1 \) and three distinct points \( Q_1, Q_2, Q_3 \in \mathbb{P}^1 \), show that there is a unique isomorphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( f(P_i) = Q_i \) for \( i = 1, 2, 3 \).

Answer:  
(i) Clearly, a morphism \( f : \mathbb{A}^1 \to \mathbb{A}^1 \) must be a polynomial map. If \( f \) has degree \( d \), the equation \( f(x) = 0 \) will have \( d \) roots. Therefore, \( d = 1 \) hence \( f(x) = ax + b \).
(ii) Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \). If \( f(\infty) = \infty \), then \( f \) maps \( \mathbb{A}^1 \to \mathbb{A}^1 \) and must have the form \( f(x) = ax + b \). Otherwise, if \( f(\infty) = \lambda \), let \( g(x) = \frac{1}{x-\lambda} \). Then \( g \circ f \) is an isomorphism which maps \( \infty \) to itself hence it must have the form \( g \circ f = cx + d \).

This shows \( f(x) = \lambda + \frac{1}{cx+d} = \frac{ax+b}{cx+d} \) for \( a = c\lambda, b = d\lambda + 1 \).

(iii) By transitivity, it suffices to assume \( P_1 = 0, P_2 = 1, P_3 = \infty \). Also, let \( Q_0 = \lambda, Q_1 = \mu, Q_2 = \nu \). Assume that none of the greek letters is \( \infty \). We require
\[
\frac{b}{d} = \lambda, \quad \frac{a+b}{c+d} = \mu, \quad \frac{a}{c} = \nu.
\]

We may take
\[
a = \nu(\mu - \lambda), \quad b = \lambda(\nu - \mu), \quad c = \mu - \lambda, \quad d = \nu - \mu.
\]

If one of the greek letters is \( \infty \), say \( \nu = \infty \), and \( \mu, \lambda \neq 0 \), we may apply the transformation \( x \to \frac{1}{x} \) to reduce to the case we have already studied.

If by contrast, \( \lambda = 0 \), then the automorphism \( f(x) = \mu x \) sends the \( P \)'s to the \( Q \)'s. If \( \mu = 0 \), then we may take
\[
f(x) = -\lambda(x - 1).
\]

To prove uniqueness, assume that two morhisms \( f_1, f_2 \) have been constructed. The inverse \( f = f_1 \circ f_2^{-1} \) has 3 fixed points \( P_1, P_2, P_3 \). We may assume that these
fixed points are $0, 1, \infty$. Since
\[ f(x) = \frac{ax + b}{cx + d} \]
and
\[ f(0) = 0, f(1) = 1, f(\infty) = \infty \]
we conclude
\[ b = 0, a + b = c + d, c = 0 \implies f = \text{id} \implies f_1 = f_2. \]

\[ \square \]

**Problem 4.** (i) Four points in $\mathbb{P}^2$ are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if $p_1, \ldots, p_4$ are points in general position, there exists a linear change of coordinates $T : \mathbb{P}^2 \to \mathbb{P}^2$ with
\[ T([1 : 0 : 0]) = p_1, \quad T([0 : 1 : 0]) = p_2, \quad T([0 : 0 : 1]) = p_3, \quad T([1 : 1 : 1]) = p_4. \]

(ii) Given five distinct points in $\mathbb{P}^2$, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part (i) to assume that four of the points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$.

(iii) Deduce that two distinct irreducible conics in $\mathbb{P}^2$ cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

**Answer:**  
(i) Let $p_i = [a_i : b_i : c_i]$ for $1 \leq i \leq 4$. Define
\[ A = \begin{pmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \beta a_2 & \beta b_2 & \beta c_2 \\ \gamma a_3 & \gamma b_3 & \gamma c_3 \end{pmatrix}, \]
where $\alpha, \beta, \gamma$ will be specified later. In fact, we will require that $\alpha, \beta, \gamma$ solve the system
\[ \alpha a_1 + \beta b_1 + \gamma c_1 = a_4, \]
\[ \alpha a_2 + \beta b_2 + \gamma c_2 = b_4, \]
\[ \alpha a_3 + \beta b_3 + \gamma c_3 = c_4. \]
A solution exists since the matrix of coefficients
\[ B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \]
is invertible. Indeed, the rows of $B$ are independent. Otherwise, a nontrivial linear relation between the rows would give a line on which the points $p_1, p_2, p_3$ lie. Thus $B$ is invertible. Now, the system above has the solution

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = B^{-1} \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}. $$

Note that the same argument shows that $A$ is invertible. Let

$$S: \mathbb{P}^2 \to \mathbb{P}^2$$

be the linear transformation defined by $A$. Then $S$ is invertible. A direct computation shows

$$S([1 : 0 : 0]) = p_1, S([0 : 1 : 0]) = p_2, S([0 : 0 : 1]) = p_3, S([1 : 1 : 1]) = p_4.$$ 

The proof is completed letting $T$ be the inverse of $S$.

(ii) After a linear change of coordinates, we may assume that the five points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$ and $[u : v : w]$. The equation $f(x, y, z)$ of any conic passing through the first three points can’t contain $x^2, y^2, z^2$, so

$$f(x, y, z) = ayz + bxz + cxy.$$ 

The remaining two points impose the conditions

$$a + b + c = avw + bw + cuv = 0.$$ 

Letting

$$A = \begin{pmatrix} 1 & 1 & 1 \\ vw & uw & uv \end{pmatrix},$$

we see that $(a, b, c)$ must be in the null space of $A$. The rank of $A$ is 2 (the rank cannot be 1 since then the rows would be proportional, hence $u = v = w$ which is not allowed). Therefore, the null space of this matrix is one dimensional, hence the conic passing through the 5 points is unique. The conic cannot be reducible since then it would be union of two lines. One of the lines would have to contain 3 points but that contradicts the general position assumption.

(iii) We first show that a line and a conic intersect in at most 2 points. Indeed, changing coordinates, we may assume that the conic $C$ is

$$xz = y^2.$$ 

Let $L$ be the line

$$\alpha x + \beta y + \gamma z = 0.$$
If \( \beta = 0 \), it is clear that \( C \) and \( L \) intersect at the points

\[
[-\gamma : \pm \sqrt{\alpha \gamma} : \alpha].
\]

If \( \beta \neq 0 \), then

\[
y = -\frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z
\]

which gives

\[
xz = \frac{1}{\beta^2}(\alpha x + \gamma z)^2.
\]

If \( x = 0 \), then \( y = z = 0 \) which is not allowed. Therefore, we may assume \( x \neq 0 \).

Dividing by \( x^2 \) we obtain the quadratic equation in \( \frac{z}{x} \):

\[
\frac{\gamma^2}{\beta^2} \left( \frac{z}{x} \right)^2 + \left( \frac{2\alpha \gamma}{\beta^2} - 1 \right) \frac{z}{x} + \frac{\alpha^2}{\beta^2} = 0.
\]

Letting \( \lambda_1, \lambda_2 \) be the solutions of this equation, we see that the intersection points are

\[
\left[ 1 : -\frac{\alpha}{\beta} - \frac{\gamma}{\beta} \lambda_i : \lambda_i \right].
\]

By the above, a conic and a line cannot intersect in 3 points. Therefore, any 5 points on an irreducible conic are in general position. Now, the claim is evident by (ii).

\[\Box\]

**Problem 5.** (Twisted curves and complete intersections.)

A variety \( Y \) of dimension \( r \) in \( \mathbb{P}^n \) is a strict complete intersection if \( I(Y) \) can be generated by \( n - r \) elements. \( Y \) is a set-theoretic complete intersection if \( Y \) can be written as the intersection of \( n - r \) hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.

(ii) Show that the twisted cubic \( T \) in \( \mathbb{P}^3 \) can be written as the set-theoretic intersection of the quadric and the cubic

\[
Q = \mathcal{Z}(y^2 - xz), \quad C = \mathcal{Z}(z^3 + x^2 - 2yzw).
\]

In particular, \( T \) is a set theoretic complete intersection.

(iii) Show that the twisted cubic in \( \mathbb{P}^3 \) is the intersections of the three quadrics

\[
Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xt - yz), Q_3 = \mathcal{Z}(yt - z^2).
\]

Show that any two of these quadrics will not intersect in the twisted cubic.

(iv) Explain that the ideal of the twisted cubic \( I(T) \) cannot be generated by two elements, hence \( T \) is not a strict complete intersection.
Answer: (i) Let $f_1, \ldots, f_{n-r}$ be the generators of the ideal $I(Y)$ for a strict complete intersection $Y$. Then

$$Y = Z(f_1) \cap \ldots \cap Z(f_{n-r})$$

exhibits $Y$ as an intersection of $n-r$ hypersurfaces.

(ii) A point in the twisted cubic $T$ has coordinates $(t^3, t^2s, ts^2, s^3)$ so it clearly satisfies the equations of $Q$ and $C$. Conversely, pick a point in the intersection of $Q$ and $C$. We compute

$$(xw - yz)^2 = x^2w^2 + y^2z^2 - 2xyzw = x(xw^2 + z^3 - 2yzw) = 0 \implies xw = yz$$

$$(yw - z^2)^2 = y^2w^2 + z^4 - 2ywz^2 = xzw^2 + z^4 - 2ywz^2 = z(xw^2 + z^3 - 2yzw) = 0 \implies yw = z^2.$$  

By (iii), we know that such a point belongs to the twisted cubic.

(iii) The twisted cubic is given by

$$[x : y : z : t] = [a^3 : a^2b : ab^2 : b^3]$$

for some $[a : b] \in \mathbb{P}^1$. Clearly, points of this type satisfy the equations

$$xz = y^2, xt = yz, yt = z^2,$$

so the twisted cubic is contained in the intersection $Q_1 \cap Q_2 \cap Q_3$.

Conversely, given a point $p = [x : y : z : t]$ in the intersection of the three quadrics $Q_1, Q_2, Q_3$, set

$$[a : b] = [x : y]$$

if $x \neq 0$. Otherwise, if $x = 0$, then $y = z = 0$ and set $[a : b] = [0 : 1]$. We claim

$$[x : y : z : t] = [a^3 : a^2b : ab^2 : b^3]$$

e.g. $p$ lies on the twisted cubic. This is clear for the case $x = 0$. If $x \neq 0$, then $a \neq 0$, and from the first and third equations we have

$$y = x \cdot \frac{b}{a}.$$  

Using the first and second equations, we solve

$$z = x \cdot \frac{b^2}{a^2}, t = x \cdot \frac{b^3}{a^3}.$$  

Then,

$$[x : y : z : t] = \left[ x : \frac{b}{a} : x \frac{b^2}{a^2} : x \frac{b^3}{a^3} \right] = [a^3 : a^2b : ab^2 : b^3].$$

Finally, note that $[0 : 0 : 1 : 0] \in Q_1 \cap Q_2$ but not in $Q_3$; $[0 : 1 : 0 : 0] \in Q_2 \cap Q_3$ but not in $Q_1$; $[1 : 0 : 0 : 1] \in Q_3 \cap Q_2$ but not in $Q_1$. So any of the two quadrics can’t generate the twisted cubic.
(iv) It is clear that $I(T)$ cannot contain linear elements $ax + by + cz + dw$ because then $T$ would be contained in a hyperplane $ax + by + cz + dw = 0$ which would mean

$$as^3 + bs^2t + cst^2 + ds^3 = 0$$

for all $s, t$ which clearly is impossible. Moreover, we have seen in (iii) that the ideal of $T$ contains 3 independent elements of degree 2. Since the space of homogeneous elements in $I(T)$ of degree 2 is 3 dimensional, any set of generators for $I(T)$ has to have at least 3 elements.

□

**Problem 6.** Let $Q$ be a quadric in $\mathbb{P}^2$. Show that $Q \cong \mathbb{P}^1$ but $S(Q) \not\cong S(\mathbb{P}^1)$.

**Answer:** After changing coordinates, we may assume that the quadric takes the form $Q = \{xy = z^2\}$. This was seen to be isomorphic to $\mathbb{P}^1$ in class via the map $[s : t] \mapsto [s^2 : t^2 : st]$. The coordinate ring $S(\mathbb{P}^1) = \mathbb{C}[s, t]$ and $S(Q) = \mathbb{C}[x, y, z]/(xy - z^2)$. Assuming that the two rings are isomorphic, consider the maximal ideal $\mathfrak{m} = (x, y, z)$ in $\mathbb{C}[x, y, z]/(xy - z^2)$. It corresponds to a maximal ideal in $\mathbb{C}[s, t]$ so it must be of the form $(s - a, t - b)$, hence generated by two elements. Thus $\mathfrak{m}$ is generated by 2 elements $f, g$ in $\mathbb{C}[x, y, z]/(xy - z^2)$, which means

$$(x, y, z) = (f, g, xy - z^2)$$

in $\mathbb{C}[x, y, z]$. By looking at the part of degree 1 we obtain that $(x, y, z)$ is a vector space isomorphic to $\langle f^{(1)}, g^{(1)} \rangle$. Comparing dimensions, we obtain a contradiction.  □

**Problem 7.** (Introduction to moduli theory.) Show that for any 3 lines $L_1, L_2, L_3$ in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric $Q$ and not on the polynomial defining it. Let us denote this point by $p_Q$. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subspace

$$H_L \subset \mathbb{P}^9$$

such that

$L \subset Q$ iff and only if $p_Q \in H_L$.  

(iii) Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that $L_1, L_2, L_3$ are contained in a quadric $Q$.

(iv) Explain (briefly) that if $L_1, L_2, L_3$ are disjoint lines, then $Q$ can be assumed to be irreducible.

**Answer:**

(i) Consider a homogeneous degree 3 polynomial

$$F(X_0, X_1, X_2, X_3) = a_0 X_0^2 + a_1 X_0 X_1 + \cdots + a_{10} X_3^2.$$  

This has 4 square terms and $\binom{4}{2} = 6$ mixed terms, i.e. 10 terms in total. We associate the quadric $Q := Z(F)$ the point

$$p_Q = [a_0 : a_1 : \ldots : a_{10}] \in \mathbb{P}^9.$$  

Note that this association is independent of the defining equation of the quadric. Indeed, if $F$ and $G$ define the same quadric, then by the Nullstellensatz it follows that $G = cF$ for some constant $c \neq 0$. But then the point $p_Q$ does not change, since the coefficients $a_i$ are considered projectively. Finally, this association is clearly bijective, since for any $p_Q \in \mathbb{P}^9$ we can recover the equation of the quadric $Q$ (up to scalars).

(ii) Assume that the line $L$ is given by $X_0 = X_1 = 0$. Then

$$L \subset Q \cong F(0, 0, X_2, X_3) = 0.$$  

Since

$$F(0, 0, X_2, X_3) = a_8 X_2^2 + a_9 X_2 X_3 + a_{10} X_3^2,$$

we obtain

$$a_8 = a_9 = a_{10} = 0.$$  

These equations define a codimension 3 projective space $H_L \subset \mathbb{P}^9$ such that

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$  

If $L'$ is any line and $Q'$ is a quadric, we can change coordinates linearly so that $L'$ becomes the line $L': X_0 = X_1 = 0$. After the coordinate change, $Q'$ is mapped to another quadric $Q$ and the coefficients of $Q$ are linear combinations of coefficients of $Q'$. From above

$$L \subset Q \text{ if and only if } p_Q \in H_L.$$
Note that

\[ L' \subset Q' \text{ if and only if } L \subset Q. \]

Therefore there exists a codimension 3 projective space \( H_L' \subset \mathbb{P}^9 \), the transformation of \( H_L \) via the change of coordinates, such that

\[ L' \subset Q' \text{ if and only if } p_{Q'} \in H_L'. \]

(iii) The codimension 3 subspaces \( H_L \) correspond to 7 dimensional linear subspaces \( A_L \subset \mathbb{A}^{10} \) cut out by the same 3 linear equations. Now,

\[ A_{L_1} \cap A_{L_2} \cap A_{L_3} \]

is described by 9 linear equations in \( \mathbb{A}^{10} \). Such a system of equations must have a nontrivial solution \( p \). This solution \( p \) considered projectively satisfies

\[ p \in H_{L_1} \cap H_{L_2} \cap H_{L_3}. \]

Now, \( p \) determines a quadric \( Q \) with \( p = p_{Q} \). It follows by (iii) that

\[ L_1 \cap L_2 \cap L_3 \subset Q. \]

(iv) If the quadric \( Q \) is reducible, then \( Q \) is the union of two hyperplanes \( H_1 \) and \( H_2 \). If \( L_1, L_2 \) and \( L_3 \) are 3 disjoint lines and

\[ L_1 \cup L_2 \cup L_3 \subset Q, \]

there should be two lines in the same hyperplane \( H_1 \cong \mathbb{P}^2 \) or \( H_2 \cong \mathbb{P}^2 \). But two lines on \( \mathbb{P}^2 \) always intersect in 1 point. This is a contradiction. Therefore we can assume \( Q \) is irreducible.

\[ \Box \]